Portfolio Optimization under Local-Stochastic Volatility: Coefficient Taylor Series Approximations & Implied Sharpe Ratio

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June 19, 2015; revised March 2016

Abstract

We study the finite horizon Merton portfolio optimization problem in a general local-stochastic volatility setting. Using model coefficient expansion techniques, we derive approximations for the both the value function and the optimal investment strategy. We also analyze the 'implied Sharpe ratio' and derive a series approximation for this quantity. The zeroth-order approximation of the value function and optimal investment strategy correspond to those obtained by Merton (1969) when the risky asset follows a geometric Brownian motion. The first-order correction of the value function can, for general utility functions, be expressed as a differential operator acting on the zeroth-order term. For power utility functions, higher order terms can also be computed as a differential operator acting on the zeroth-order term. While our approximations are derived formally, we give a rigorous accuracy bound for the higher order approximations in this case in pure stochastic volatility models. A number of examples are provided in order to demonstrate numerically the accuracy of our approximations.

1 Introduction

The continuous time portfolio optimization problem was first studied by Merton (1969), where he considers a market that contains a riskless bond, which grows at a fixed deterministic rate, and multiple risky assets, each of which is modeled as a geometric Brownian motion with constant drift and constant volatility. In this setting, Merton obtains an explicit expression for the value function and optimal investment strategy of an investor who wishes to maximize expected utility when the utility function has certain specific forms. However, much empirical evidence suggests that volatility is stochastic and is driven by both local and auxiliary factors, and so it is natural to ask how an investor would change his investment strategy in the presence of stochastic volatility.

There have been a number of studies in this direction, a few of which, we now mention. Darius (2005) studies the finite horizon optimal investment problem in a CEV local volatility model. Chacko and Viceira (2005) examine the infinite-horizon optimal investment problem in a Heston-like stochastic volatility model. While both studies provide an explicit expression for an investor's value function and optimal investment strategy, the results are specific to the models studied in these two papers and for power utility functions.

Approximation methods, which have been extensively used for option pricing and related problems, have been adapted for the portfolio selection problem, allowing for a wider class of volatility

*Department of Applied Mathematics, University of Washington, Seattle, WA, USA. Work partially supported by NSF grant DMS-0739195.

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models and utility functions. The Merton problem for power utilities under *fast mean-reverting* stochastic volatility was analyzed by asymptotic methods in (Fouque et al., 2000, Chapter 10), and the related partial hedging stochastic control problem in Jonsson and Sircar (2002b,a) using asymptotic analysis for the dual problem. More recently, Fouque et al. (2013) consider a general class of multiscale stochastic volatility models and general utility functions. Here, volatility is driven by one fast-varying and one slow-varying factor. The separation of time-scales allows the authors to obtain explicit approximations for the investor's value function and optimal control, by combining singular and regular perturbation methods on the primal problem. These methods were previously developed to obtain explicit price approximations for various financial derivatives, as described in the book Fouque et al. (2011).

Here we study the Merton problem in a general local-stochastic volatility (LSV) setting. The LSV setting encompasses both local volatility models (e.g., CEV and quadratic) and stochastic volatility models (e.g., Heston and Hull-White) as well as models that combine local and auxiliary factors of volatility (e.g., SABR and λ-SABR). As explicit expressions for the value function and optimal investment strategy are not available in this very general setting, we focus on obtaining approximations for these quantities. Specifically, we will obtain an approximation for the solution of a nonlinear Hamilton-Jacobi-Bellman partial differential equation (HJB PDE) by expanding the PDE coefficients in a Taylor series. The Taylor series expansion method was initially developed in Pagliarani and Pascucci (2012) to solve linear pricing PDEs under local volatility models, and is closely related to the classical parametrix method (see, for instance, Corielli et al. (2010) for applications in finance). The method was later extended in Lorig et al. (2015b) to include more general polynomial expansions and to handle multidimensional diffusions. Additionally, the technique has been applied to models with jumps; see Pagliarani et al. (2013), Lorig et al. (2015c) and Lorig et al. (2014). We remark that the PDEs that arise in no-arbitrage pricing theory are linear, whereas the HJB PDE we consider here is fully nonlinear.

The rest of the paper proceeds as follows. In Section 2, we introduce a general class of local-stochastic volatility models, define a representative investor's value function and write the associated HJB PDE. Section 3 presents the first order approximation and formulas for the principal LSV correction to the value function and the optimal investment strategy, which are derived formally. Motivated by the notion of Black-Scholes implied volatility, we also develop the notion of implied Sharpe ratio, that provides a greater intuition about the resulting formulas, which are summarized in Section 3.7. We discuss higher order terms in Section 4, and show that power utilities are particularly amenable to obtaining explicit formulas for further terms in the approximation. In Section 5, we provide explicit results for power utility. In particular, we derive rigorous error bounds for the value function in a stochastic volatility setting. In Section 6, we provide two numerical examples, illustrating the accuracy and versatility of our approximation method. Section 7 concludes.

2 Merton Problem under Local-Stochastic Volatility

We consider a local-stochastic volatility model for a risky asset S:

$$\begin{aligned} \frac{\mathrm{d}S_t}{S_t} &= \tilde{\mu}(S_t, Y_t)\,\mathrm{d}t + \tilde{\sigma}(S_t, Y_t)\,\mathrm{d}B_t^{(1)} \\ \mathrm{d}Y_t &= \tilde{c}(S_t, Y_t)\,\mathrm{d}t + \tilde{\beta}(S_t, Y_t)\,\mathrm{d}B_t^{(2)}, \end{aligned} \tag{1}$$

where $B^{(1)}$ and $B^{(2)}$ are standard Brownian motions under a probability measure $\mathbb{P}$ with correlation coefficient $\rho \in (-1, 1)$: $\mathbb{E}\{\mathrm{d}B_t^{(1)}\,\mathrm{d}B_t^{(2)}\} = \rho\,\mathrm{d}t$. The log price process $X = \log S$ is, by Itô's formula, described by the following :

$$
\begin{aligned}
\mathrm{d}X_t &= b(X_t, Y_t)\,\mathrm{d}t + \sigma(X_t, Y_t)\,\mathrm{d}B_t^{(1)} \\
\mathrm{d}Y_t &= c(X_t, Y_t)\,\mathrm{d}t + \beta(X_t, Y_t)\,\mathrm{d}B_t^{(2)},
\end{aligned} \tag{2}
$$

where $\sigma(X_t, Y_t) = \tilde{\sigma}(e^{X_t}, Y_t)$, and similarly (μ, c, β) from $(\tilde{\mu}, \tilde{c}, \tilde{\beta})$, and we have defined

$$
b(X_t, Y_t) = \mu(X_t, Y_t) - \frac{1}{2}\sigma^2(X_t, Y_t).
$$

The model coefficient functions (μ, σ, c, β) are smooth functions of (x, y) and are such that the Markovian system (2) admits a unique strong solution.

2.1 Utility Maximization and HJB Equation

We denote by W the wealth process of an investor who invests π_t units of currency in S at time t and invests $(W_t - \pi_t)$ units of currency in a riskless money market account. For simplicity, we assume that the risk-free rate of interest is zero, and so the wealth process W satisfies

$$
\mathrm{d}W_t = \frac{\pi_t}{S_t}\,\mathrm{d}S_t = \pi_t \mu(X_t, Y_t)\,\mathrm{d}t + \pi_t \sigma(X_t, Y_t)\,\mathrm{d}B_t^{(1)}.
$$

The investor acts to maximize the expected utility of portfolio value, or wealth, at a fixed finite time horizon T: $\mathbb{E}\left\{U(W_T)\right\}$, where $U : \mathbb{R}_+ \to \mathbb{R}$ is a smooth, increasing and strictly concave utility function satisfying the "usual conditions" $U'(0^+) = \infty$ and $U'(\infty) = 0$.

We define the investor's *value function* V by

$$
V(t, x, y, w) := \sup_{\pi \in \Pi} \mathbb{E}\{U(W_T) \mid X_t = x, Y_t = y, W_t = w\}, \tag{3}
$$

where Π is the set of *admissible strategies* π, which are non-anticipating and satisfy

$$
\mathbb{E}\left\{ \int_0^T \pi_t^2 \sigma^2(X_t, Y_t)\,\mathrm{d}t \right\} < \infty.
$$

and $W_t \geq 0$ a.s.

We assume that $V \in C^{1,2,2,2}([0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+)$. The Hamilton-Jacobi-Bellman partial differential equation (HJB-PDE) associated with the stochastic control problem (3) is

$$
\left(\frac{\partial}{\partial t} + \mathcal{A}\right) V + \max_{\pi \in \mathbb{R}} \mathcal{A}^\pi V = 0, \qquad V(T, x, y, w) = U(w), \tag{4}
$$

where the operators $\mathcal{A}$ and $\mathcal{A}^\pi$ are given by

$$
\begin{aligned}
\mathcal{A} &= \tfrac{1}{2}\sigma^2(x,y)\frac{\partial^2}{\partial x^2} + \rho\sigma(x,y)\beta(x,y)\frac{\partial^2}{\partial x \partial y} + \tfrac{1}{2}\beta^2(x,y)\frac{\partial^2}{\partial y^2} + b(x,y)\frac{\partial}{\partial x} + c(x,y)\frac{\partial}{\partial y}, \\
\mathcal{A}^\pi &= \frac{1}{2}\pi^2\sigma^2(x,y)\frac{\partial^2}{\partial w^2} + \pi\left(\sigma^2(x,y)\frac{\partial^2}{\partial x \partial w} + \rho\sigma(x,y)\beta(x,y)\frac{\partial^2}{\partial y \partial w} + \mu(x,y)\frac{\partial}{\partial w}\right).
\end{aligned} \tag{5}
$$

We refer to the books Fleming and Soner (1993) and Pham (2009) for technical details. The regularity assumption is a standard one under which verification theorems for stochastic control problems are proved. See, for instance, (Pham, 2009, Theorem 3.5.2). While the PDE (4) is fully nonlinear, the approximations constructed in this paper will have as their principal term the solution of the complete markets Merton problem, for which regularity is well-established, by Legendre transformation to a linear PDE problem. Thus by staying close in a certain sense to a case with a classical solution, we do not deal with viscosity solutions.

The optimal strategy $\pi^* = \operatorname{argmax}_\pi \mathcal{A}^\pi V$ is given (in feedback form) by

$$\pi^* = -\frac{\left(\sigma^2(x,y)V_{xw} + \rho\sigma(x,y)\beta(x,y)V_{yw} + \mu(x,y)V_w\right)}{\sigma^2(x,y)V_{ww}}, \tag{6}$$

where subscripts indicate partial derivatives.

Inserting the optimal strategy π^* into the HJB-PDE (4) yields

$$\left(\frac{\partial}{\partial t} + \mathcal{A}\right)V + \mathcal{N}(V) = 0, \qquad V(T,x,y,w) = U(w), \tag{7}$$

where $\mathcal{N}(V)$ is a nonlinear term, which is given by

$$\mathcal{N}(V) = -\frac{\left(\sigma(x,y)V_{xw} + \rho\beta(x,y)V_{yw} + \lambda(x,y)V_w\right)^2}{2V_{ww}}, \tag{8}$$

and we have introduced the *Sharpe ratio*

$$\lambda(x,y) = \frac{\mu(x,y)}{\sigma(x,y)}.$$

2.2 Constant Parameter Merton Problem

We review and introduce notation that will be used later for the *constant parameter* Merton problem, that is, when $\tilde{\mu}$ and $\tilde{\sigma}$ in (1) are constant, and so therefore μ and σ are constant and the stock price S follows the geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dB_t^{(1)}.$$

Then the Merton value function $M(t,w;\lambda)$ for the investment problem for this stock, whose constant Sharpe ratio is $\lambda = \mu/\sigma$, is the unique smooth solution of the HJB PDE problem

$$M_t - \tfrac{1}{2}\lambda^2\frac{M_w^2}{M_{ww}} = 0, \qquad M(T,w) = U(w), \tag{9}$$

on $t < T$ and $w > 0$. Smoothness of M given a smooth utility function U (as assumed above), as well as differentiability of M in λ is easily established by the Legendre transform, which converts (9) into a linear constant coefficient parabolic PDE problem for the dual. Regularity results for the latter problem are standard.

It is convenient to introduce the Merton risk tolerance function

$$R(t,w;\lambda) := -\frac{M_w}{M_{ww}}(t,w;\lambda), \tag{10}$$

and the operator notation

$$\mathcal{D}_k := \left(R(t,w;\lambda)\right)^k \frac{\partial^k}{\partial w^k}, \quad k = 1, 2, \cdots. \tag{11}$$

We recall also the Vega-Gamma relationship taken from (Fouque et al., 2013, Lemma 3.2):

Lemma 2.1. *The Merton value function* $M(t,w;\lambda)$ *satisfies the "Vega-Gamma" relation*

$$\frac{\partial M}{\partial \lambda} = -(T-t)\lambda \mathcal{D}_2 M,$$

where $\mathcal{D}_2$ *is defined in* (11).

Thus the derivative of the value function with respect to the Sharpe ratio (analogous to an option price's derivative with respect to volatility, its Vega) is proportional to its negative "second derivative" $\mathcal{D}_2 M$ (which is analogous to the option price's second derivative with respect to the stock price, its Gamma). This result will be used repeatedly in deriving the implied Sharpe ratio in Section 3.5 and the approximation to the optimal portfolio in Section 3.6.

3 Coefficient Expansion & First Order Approximation

For general $\{c, \beta, \mu, \sigma, \lambda\}$ and U, there is no closed form solution of (7). The goal of this section is to formally derive series approximations for the value function

$$V = V^{(0)} + V^{(1)} + V^{(2)} + \cdots,$$

and the optimal investment strategy

$$\pi^* = \pi_0^* + \pi_1^* + \pi_2^* + \cdots,$$

using model coefficient (Taylor series) expansions. This approach is developed for the linear European option pricing problem in a general LSV setting in Lorig et al. (2015b), where explicit approximations for option prices and implied volatilities are obtained by expanding the coefficients of the underlying diffusion as a Taylor series. Note that, here the HJB-PDE (7) is fully nonlinear. Our first order approximation formulas are summarized in Section 3.7.

3.1 Coefficient Polynomial Expansions

We begin by fixing a point $(\bar{x}, \bar{y}) \in \mathbb{R}^2$. For any function $\chi(x,y)$ that is analytic in a neighborhood of $(\bar{x}, \bar{y})$, we define the following family of functions indexed by $a \in [0,1]$:

$$\chi^a(x,y) := \sum_{n=0}^{\infty} a^n \chi_n(x,y), \tag{12}$$

where

$$\chi_n(x,y) := \sum_{k=0}^{n} \chi_{n-k,k} \cdot (x-\bar{x})^{n-k}(y-\bar{y})^k, \qquad \chi_{n-k,k} := \frac{1}{(n-k)!k!}\partial_x^{n-k}\partial_y^k \chi(\bar{x},\bar{y}),$$

and we note that $\chi_0 = \chi_{0,0} = \chi(\bar{x}, \bar{y})$ is a constant. Observe that $\chi^a|_{a=1}$ is the Taylor series of χ about the point $(\bar{x}, \bar{y})$. Here, a is an *accounting parameter* that will be used to identify successive terms of our approximation.

In the PDE (7), we will replace each of the coefficient functions

$$\chi \in \{\mu, c, \sigma^2, \beta^2, \lambda^2, \sigma\beta, \beta\lambda\}$$

by χ^a, for some $a, (\bar{x}, \bar{y})$, and then use the series expansion (12) for χ^a. Another way of saying this is that we assume the coefficients are of the form

$$\chi\Big(\bar{x} + a(x - \bar{x}), \bar{y} + a(y - \bar{y})\Big),$$

whose exact Taylor series is given by (12), and we are interested in the case when $a = 1$.

Consider now the following family of HJB-PDE problems

$$\left(\frac{\partial}{\partial t} + \mathcal{A}^a\right) V^a + \mathcal{N}^a(V^a) = 0, \qquad V^a(T, x, y, w) = U(w), \tag{13}$$

where, for $a \in [0, 1]$, $\mathcal{A}^a$ and $\mathcal{N}^a(\cdot)$ are obtained from $\mathcal{A}$ and $\mathcal{N}(\cdot)$ in (5) and (8) by making the change

$$\{\mu, c, \sigma^2, \beta^2, \lambda^2, \sigma\beta, \beta\lambda\} \mapsto \{\mu^a, c^a, (\sigma^2)^a, (\beta^2)^a, (\lambda^2)^a, (\sigma\beta)^a, (\beta\lambda)^a\}.$$

The linear operator in the PDE (13) can therefore be written as

$$\mathcal{A}^a = \sum_{n=0}^{\infty} a^n \mathcal{A}_n,$$

where we define

$$\mathcal{A}_n := (\tfrac{1}{2}\sigma^2)_n(x, y)\frac{\partial^2}{\partial x^2} + (\rho\sigma\beta)_n(x, y)\frac{\partial^2}{\partial x \partial y} + (\tfrac{1}{2}\beta^2)_n(x, y)\frac{\partial^2}{\partial y^2} + b_n(x, y)\frac{\partial}{\partial x} + c_n(x, y)\frac{\partial}{\partial y}, \tag{14}$$

and the expansion of the nonlinear term is a more involved computation.

We construct a series approximation for the function V^a as a power series in a:

$$V^a(t, x, y, w) = \sum_{n=0}^{\infty} a^n V^{(n)}(t, x, y, w). \tag{15}$$

Note that the functions $V^{(n)}$ are not constrained to be polynomials in (x, y, w), and in general they will not be. Our approximate solution to (7), which is the problem of interest, will then follow by setting $a = 1$.

3.2 Zeroth & First Order Approximations

We insert (15) into (13) and collect terms of like powers of a. At lowest order we obtain

$$\left(\frac{\partial}{\partial t} + \mathcal{A}_0\right) V^{(0)} - \frac{\left(\sigma_0 V^{(0)}_{xw} + \rho\beta_0 V^{(0)}_{yw} + \lambda_0 V^{(0)}_w\right)^2}{2V^{(0)}_{ww}} = 0, \qquad V^{(0)}(T, x, y, w) = U(w), \tag{16}$$

where the linear operator $\mathcal{A}_0$, found from (14), has constant coefficients:

$$\mathcal{A}_0 = \tfrac{1}{2}\sigma_0^2\frac{\partial^2}{\partial x^2} + \rho\sigma_0\beta_0\frac{\partial^2}{\partial x \partial y} + \tfrac{1}{2}\beta_0^2\frac{\partial^2}{\partial y^2} + b_0\frac{\partial}{\partial x} + c_0\frac{\partial}{\partial y}. \tag{17}$$

As a consequence, the solution of (16) is independent of x and y: $V^{(0)} = V^{(0)}(t, w)$, and we have

$$V^{(0)}_t - \tfrac{1}{2}\lambda_0^2\frac{\left(V^{(0)}_w\right)^2}{V^{(0)}_{ww}} = 0, \qquad V^{(0)}(T, w) = U(w). \tag{18}$$

We observe that (18) is the same as the PDE problem (9) that arises when solving the Merton problem assuming the underlying stock has a constant drift $\mu_0 = \mu(\bar{x}, \bar{y})$, diffusion coefficient $\sigma_0 = \sigma(\bar{x}, \bar{y})$ and so constant Sharpe ratio $\lambda_0 = \lambda(\bar{x}, \bar{y}) = \mu(\bar{x}, \bar{y})/\sigma(\bar{x}, \bar{y})$. Therefore, we have

$$V^{(0)}(t, w) = M(t, w; \lambda_0).$$

The PDE (18) can be solved either analytically (for certain utility functions U), or numerically.

Recall the definition of the risk tolerance function in (10) and the operators $\mathcal{D}_k$ in (11), where now we take in those formulas the Sharpe ratio λ_0:

$$R(t, w; \lambda_0) = -\frac{V_w^{(0)}}{V_{ww}^{(0)}}(t, w; \lambda), \qquad \mathcal{D}_k = (R(t, w; \lambda_0))^k \frac{\partial^k}{\partial w^k}, \quad k = 1, 2, \cdots.$$

Proceeding to the order a terms in (13), we obtain

$$\left(\frac{\partial}{\partial t} + \mathcal{A}_0\right) V^{(1)} + \tfrac{1}{2}\lambda_0^2 \mathcal{D}_2 V^{(1)} + \lambda_0^2 \mathcal{D}_1 V^{(1)} + \rho\beta_0\lambda_0 \mathcal{D}_1 \frac{\partial}{\partial y} V^{(1)} + \mu_0 \mathcal{D}_1 \frac{\partial}{\partial x} V^{(1)} = -(\tfrac{1}{2}\lambda^2)_1 \mathcal{D}_1 V^{(0)}.$$

We can re-write this more compactly as

$$\left(\frac{\partial}{\partial t} + \mathcal{A}_0 + \mathcal{B}_0\right) V^{(1)} + H_1 = 0, \qquad V^{(1)}(T, x, y, w) = 0, \tag{19}$$

where the linear operator $\mathcal{B}_0$ and the source term H_1 are given by

$$\mathcal{B}_0 = \tfrac{1}{2}\lambda_0^2 \mathcal{D}_2 + \lambda_0^2 \mathcal{D}_1 + \rho\beta_0\lambda_0 \mathcal{D}_1 \frac{\partial}{\partial y} + \mu_0 \mathcal{D}_1 \frac{\partial}{\partial x}, \tag{20}$$

$$H_1(t, x, y, w) = (\tfrac{1}{2}\lambda^2)_1(x, y) \mathcal{D}_1 V^{(0)}(t, w). \tag{21}$$

We now proceed to give the solution of the linear PDE problem (19).

3.3 Transformation to Constant Coefficient PDEs

First, we apply a change of variable such that $V^{(1)}$ can be found by solving a linear PDE with constant coefficients. We begin with the following lemma.

Lemma 3.1. *Let $V^{(0)}$ be the solution of* (18) *and let $\mathcal{A}_0$ and $\mathcal{B}_0$ be as given in* (14) *and* (21)*, respectively. Then $V^{(0)}$ satisfies the following PDE problem*

$$\left(\frac{\partial}{\partial t} + \mathcal{A}_0 + \mathcal{B}_0\right) V^{(0)} = 0, \qquad V^{(0)}(T, w) = U(w). \tag{22}$$

Proof. This follows directly from observing that the nonlinear term in (18) can be written

$$\frac{\left(V_w^{(0)}\right)^2}{V_{ww}^{(0)}} = \left(-\frac{V_w^{(0)}}{V_{ww}^{(0)}}\right)^2 V_{ww}^{(0)} = \mathcal{D}_2 V^{(0)}, \quad \text{or} \quad \frac{\left(V_w^{(0)}\right)^2}{V_{ww}^{(0)}} = -\left(-\frac{V_w^{(0)}}{V_{ww}^{(0)}}\right) V_w^{(0)} = -\mathcal{D}_1 V^{(0)}. \tag{23}$$

Therefore, from (18), we have

$$\left(\frac{\partial}{\partial t} + \tfrac{1}{2}\lambda_0^2 \mathcal{D}_2 + \lambda_0^2 \mathcal{D}_1\right) V^{(0)} = 0,$$

and (22) follows from the fact that $V^{(0)}$ does not depend on (x, y), while $\mathcal{A}_0$ and the last two terms in the expression (20) for $\mathcal{B}_0$ take derivatives in those variables. □

Next, it will be helpful to introduce the following change of variables.

Definition 3.2. We define the co-ordinate z by the transformation

$$z(t,w) = -\log V_w^{(0)}(t,w) + \tfrac{1}{2}\lambda_0^2(T-t). \tag{24}$$

We have the following change of variables formula, as used also in (Fouque et al., 2013, Section 2.3.2).

Lemma 3.3. *For a smooth function $\widehat{V}(t,x,y,w)$, define $q(t,x,y,z)$ by*

$$\widehat{V}(t,x,y,w) = q(t,x,y,z(t,w)).$$

Then we have

$$\left(\frac{\partial}{\partial t} + \mathcal{A}_0 + \mathcal{B}_0\right)\widehat{V} = \left(\frac{\partial}{\partial t} + \mathcal{A}_0 + \mathcal{C}_0\right) q, \tag{25}$$

where the operator $\mathcal{C}_0$ is given by

$$\mathcal{C}_0 = \tfrac{1}{2}\lambda_0^2\frac{\partial^2}{\partial z^2} + \rho\beta_0\lambda_0\frac{\partial^2}{\partial y\partial z} + \mu_0\frac{\partial^2}{\partial x\partial z}. \tag{26}$$

Proof. We shall use the shorthand $R^{(0)}(t,w) = R(t,w;\lambda_0)$. From (24), we have that $z_w = 1/R^{(0)}$, and so, differentiating (30), we find

$$\widehat{V}_t = q_t - \left(\frac{V_{tw}^{(0)}}{V_w^{(0)}} + \tfrac{1}{2}\lambda_0^2\right) q_z, \qquad \mathcal{D}_1\widehat{V} = q_z, \qquad \mathcal{D}_2\widehat{V} = q_{zz} - R_w^{(0)} q_z.$$

Then, using the first expression in (23) to write the PDE (18) for $V^{(0)}$ as $V_t^{(0)} = \frac{1}{2}\lambda_0^2\mathcal{D}_2 V^{(0)}$, and differentiating this with respect to w gives

$$V_{tw}^{(0)} = \tfrac{1}{2}\lambda_0^2\left(R^{(0)}\right)^2 V_{www}^{(0)} + \lambda_0^2 R^{(0)} R_w^{(0)} V_{ww}^{(0)}.$$

But from $R^{(0)}V_{ww}^{(0)} = -V_w^{(0)}$, we have $(R^{(0)})^2 V_{www}^{(0)} = (R_w^{(0)}+1)V_w^{(0)}$, and so

$$V_{tw}^{(0)} = \tfrac{1}{2}\lambda_0^2(R_w^{(0)}+1)V_w^{(0)} - \lambda_0^2 R_w^{(0)} V_w^{(0)},$$

which gives that

$$\frac{V_{tw}^{(0)}}{V_w^{(0)}} = -\tfrac{1}{2}\lambda_0^2(R_w^{(0)}-1).$$

Therefore, we have

$$\left(\frac{\partial}{\partial t} + \tfrac{1}{2}\lambda_0^2\mathcal{D}_2 + \lambda_0^2\mathcal{D}_1\right)\widehat{V} = q_t + \left(\tfrac{1}{2}\lambda_0^2(R_w^{(0)}-1) - \tfrac{1}{2}\lambda_0^2\right) q_z + \tfrac{1}{2}\lambda_0^2\left(q_{zz} - R_w^{(0)} q_z\right) + \lambda_0^2 q_z,$$

which establishes that

$$\left(\frac{\partial}{\partial t} + \tfrac{1}{2}\lambda_0^2\mathcal{D}_2 + \lambda_0^2\mathcal{D}_1\right)\widehat{V} = \left(\frac{\partial}{\partial t} + \tfrac{1}{2}\lambda_0^2\frac{\partial^2}{\partial z^2}\right) q. \tag{27}$$

More directly, we have

$$\left(\rho\beta_0\lambda_0\mathcal{D}_1\frac{\partial}{\partial y} + \mu_0\mathcal{D}_1\frac{\partial}{\partial x}\right)\widehat{V} = \left(\rho\beta_0\lambda_0\frac{\partial^2}{\partial y\partial z} + \mu_0\frac{\partial^2}{\partial x\partial z}\right) q,$$

which, combined with (27), leads to (25). □

We define $q^{(0)}$ by $V^{(0)}(t,w) = q^{(0)}(t, z(t,w))$. Then the PDE (22) for $V^{(0)}$ is transformed to the (constant coefficient) backward heat equation for $q^{(0)}(t,z)$:

$$\left(\frac{\partial}{\partial t} + \tfrac{1}{2}\lambda_0^2 \frac{\partial^2}{\partial z^2}\right) q^{(0)} = 0, \qquad q^{(0)}(T,z) = U\left((U')^{-1}(e^{-z})\right), \tag{28}$$

but of course the transformation (24) depends on the solution $V^{(0)}$ itself. Again, as $q^{(0)}$ does not depend on (x,y), while $\mathcal{A}_0$ and the last two terms in the expression (26) for $\mathcal{C}_0$ take derivatives in those variables, we can write

$$\left(\frac{\partial}{\partial t} + \mathcal{A}_0 + \mathcal{C}_0\right) q^{(0)} = 0. \tag{29}$$

Now let $q^{(1)}$ be defined from $V^{(1)}$ by

$$V^{(1)}(t,x,y,w) = q^{(1)}(t,x,y,z(t,w)), \tag{30}$$

using the transformation (24). Then, using Lemma 3.3, we see that the PDE (19) for $V^{(1)}$, which has (t,w)-dependent coefficients through the dependence of $\mathcal{B}_0$ in (20) on $R(t,w;\lambda_0)$, is transformed to the *constant coefficient* equation for $q^{(1)}$:

$$\left(\frac{\partial}{\partial t} + \mathcal{A}_0 + \mathcal{C}_0\right) q^{(1)} + Q_1 = 0, \qquad q^{(1)}(T,x,y,z) = 0. \tag{31}$$

The source term is found from $H_1(t,x,y,w) = Q_1(t,x,y,z(t,w))$, where, from (21), we have

$$Q_1(t,x,y,z) = (\tfrac{1}{2}\lambda^2)_1(x,y) q_z^{(0)}. \tag{32}$$

3.4 Explicit expression for $V^{(1)}$

In this section, we will show that $V^{(1)}$, solution of (19), can be written as a differential operator acting on $V^{(0)}$. First, we look at the PDE problem

$$\mathcal{H}q + Q = 0, \qquad q(T,x,y,z) = 0, \tag{33}$$

where $\mathcal{H}$ is the constant coefficient linear operator

$$\mathcal{H} = \frac{\partial}{\partial t} + \mathcal{A}_0 + \mathcal{C}_0.$$

We also suppose that the source term $Q(t,x,y,z)$ is of the following special form:

$$Q(t,x,y,z) = \sum_{k,l,n} (T-t)^n (x-\bar{x})^k (y-\bar{y})^l v(t,x,y,z), \tag{34}$$

where the sum is finite and v is a solution of the homogeneous equation

$$\mathcal{H}v = 0. \tag{35}$$

We define (the commutator) $\mathcal{L}_X = [\mathcal{H}, (x-\bar{x})I]$ by

$$\mathcal{H}\left((x-\bar{x})v\right) = (x-\bar{x})\mathcal{H}v + \mathcal{L}_X v, \tag{36}$$

and so a direct calculation using the expressions (17) and (26) for $\mathcal{A}_0$ and $\mathcal{C}_0$ respectively shows that

$$\mathcal{L}_X = (\mu_0 - \tfrac{1}{2}\sigma_0^2)I + \sigma_0^2\frac{\partial}{\partial x} + \rho\sigma_0\beta_0\frac{\partial}{\partial y} + \mu_0\frac{\partial}{\partial z}, \tag{37}$$

where I is the identity operator. Similarly, defining (the commutator) $\mathcal{L}_Y = [\mathcal{H}, (y-\bar{y})I]$ by

$$\mathcal{H}\left((y-\bar{y})v\right) = (y-\bar{y})\mathcal{H}v + \mathcal{L}_Y v,$$

leads to

$$\mathcal{L}_Y = c_0 I + \beta_0^2\frac{\partial}{\partial y} + \rho\sigma_0\beta_0\frac{\partial}{\partial x} + \rho\beta_0\lambda_0\frac{\partial}{\partial z}, \tag{38}$$

We next introduce the following operators indexed by $s \in [t,T]$:

$$\mathcal{M}_X(s) = (x-\bar{x})I + (s-t)\mathcal{L}_X, \qquad \mathcal{M}_Y(s) = (y-\bar{y})I + (s-t)\mathcal{L}_Y, \tag{39}$$

Then we have the following result by construction of these operators.

Lemma 3.4. *Recall that v solves the homogeneous equation* (35). *Then*

$$\mathcal{H}\mathcal{M}_X^k(s)\mathcal{M}_Y^l(s)v = 0, \tag{40}$$

for integers k, l.

Proof. We first calculate

$$\mathcal{H}\mathcal{M}_X v = \mathcal{M}_X\mathcal{H}v + \mathcal{L}_X v - \mathcal{L}_X v + (s-t)\mathcal{H}\mathcal{L}_X v = (s-t)\mathcal{H}\mathcal{L}_X v,$$

where we have used (36). But since $\mathcal{H}$ and $\mathcal{L}_X$ are constant coefficient operators which commute, we have $\mathcal{H}\mathcal{L}_X v = \mathcal{L}_X\mathcal{H}v = 0$ using (35). Therefore, given a solution v of the homogeneous equation, $\mathcal{M}_X v$ also solves the homogeneous equation, namely $\mathcal{H}\left(\mathcal{M}_X v\right) = 0$. Iterating we have that $\mathcal{H}\mathcal{M}_X^k v = 0$ for integers k. Similarly $\mathcal{H}\mathcal{M}_Y^l v = 0$ for integers l, and so the result (40) follows. □

From this we can exploit the special structure of the source Q to obtain the following formula.

Proposition 3.5. *The solution to* (33) *where the source Q is of the form* (34) *is given by*

$$q(t,x,y,z) = \sum_{k,l,n}\int_t^T (T-s)^n\mathcal{M}_X^k(s)\mathcal{M}_Y^l(s)v(t,x,y,z)\,ds. \tag{41}$$

Proof. Due to the linearity of the problem, it suffices to consider a single term of the polynomial:

$$Q(t,x,y,z) = (T-t)^n(x-\bar{x})^k(y-\bar{y})^l v(t,x,y,z).$$

Then, we check that the solution is given by

$$q(t,x,y,z) = \int_t^T (T-s)^n\mathcal{M}_X^k(s)\mathcal{M}_Y^l(s)v(t,x,y,z)\,ds$$

by computing

$$\begin{aligned}
\mathcal{H}q &= -(T-t)^n\mathcal{M}_X^k(t)\mathcal{M}_Y^l(t)v(t,x,y,z) + \int_t^T (T-s)^n\mathcal{H}\mathcal{M}_X^k(s)\mathcal{M}_Y^l(s)v(t,x,y,z)\,ds \\
&= -(T-t)^n(x-\bar{x})^k(y-\bar{y})^l v(t,x,y,z) \\
&= -Q,
\end{aligned}$$

using Lemma 3.4 for the second term. The formula (41) in the general polynomial case follows, and clearly the zero terminal condition is satisfied by (41). □

We can now solve for the first correction in the series expansion.

Proposition 3.6. *The solution to* (31) *is given by*

$$q^{(1)}(t,x,y,z) = (T-t)\lambda_0 A(t,x,y)\, q_z^{(0)}(t,z) + \tfrac{1}{2}(T-t)^2\lambda_0 B\, q_{zz}^{(0)}(t,z), \tag{42}$$

where

$$\begin{aligned} A(t,x,y) &= \lambda_{1,0}\left[(x-\bar{x}) + \tfrac{1}{2}(T-t)(\mu_0 - \tfrac{1}{2}\sigma_0^2)\right] + \lambda_{0,1}\left[(y-\bar{y}) + \tfrac{1}{2}(T-t)c_0\right], \\ B &= \lambda_{1,0}\mu_0 + \lambda_{0,1}\rho\beta_0\lambda_0. \end{aligned} \tag{43}$$

Proof. We observe that since $q^{(0)}$ satisfies the homogeneous PDE $\mathcal{H}q^{(0)} = 0$ from (29), so does $q_z^{(0)}$, which follows from differentiating the constant coefficient PDE for $q^{(0)}$. Then applying Proposition 3.5 with $v = q_z^{(0)}$, and $n = 0, (k,l) \in \{(1,0),(0,1)\}$ and substituting the definitions (39) for $\mathcal{M}_X$ and $\mathcal{M}_Y$ leads to

$$\begin{aligned} q^{(1)}(t,x,y,z) = \Big[&(\tfrac{1}{2}\lambda^2)_{1,0}\left((T-t)(x-\bar{x}) + \tfrac{1}{2}(T-t)^2\mathcal{L}_X\right) \\ &+(\tfrac{1}{2}\lambda^2)_{0,1}\left((T-t)(y-\bar{y}) + \tfrac{1}{2}(T-t)^2\mathcal{L}_Y\right)\Big]\, q_z^{(0)}(t,z). \end{aligned}$$

Finally, substituting for $\mathcal{L}_X$ and $\mathcal{L}_Y$ from (37) and (38) and using that $q^{(0)}$ does not depend on (x,y) leads to (42). ☐

In the original variables, using (30), we have

$$V^{(1)}(t,x,y,w) = (T-t)\lambda_0 A(t,x,y)\,\mathcal{D}_1 V^{(0)}(t,w) + \tfrac{1}{2}(T-t)^2\lambda_0 B\,\mathcal{D}_1^2 V^{(0)}(t,w). \tag{44}$$

3.5 Implied Sharpe Ratio

In an analogy to implied volatility, for a fixed maturity T and utility function U, one can define the *Merton implied Sharpe ratio*[1] corresponding to value function $M(t,w;\lambda)$ of Section 2.2 as the unique positive solution $\Lambda^a(t,x,y,w)$ of

$$V^a(t,x,y,w) = M(t,w;\Lambda^a). \tag{45}$$

The existence and uniqueness of the implied Sharpe ratio follows from the fact that (i) the function M satisfies $M(t,w) \geq U(w)$, since an investor with initial wealth w can always obtain a terminal utility $U(w)$ by investing all of his money in the riskless bank account, and (ii) the function M is strictly increasing in Λ. Since a higher implied Sharpe ratio is indicative of a better investment opportunity, we are interested to know how local stochastic volatility model parameters $\{c,\beta,\mu,\sigma,\rho\}$ affect the implied Sharpe ratio.

Using our first order approximation $V^a \approx V^{(0)} + aV^{(1)}$, we look for a corresponding series approximation of the implied Sharpe ratio as

$$\Lambda^a = \Lambda^{(0)} + a\Lambda^{(1)} + \cdots .$$

Then, expanding

$$M(t,w;\Lambda) = M(t,w;\Lambda^{(0)}) + a\Lambda^{(1)} M_\lambda(t,w;\Lambda^{(0)}) + \cdots ,$$

[1]The authors thank Jean-Pierre Fouque for a number of fruitful discussions, from which the concept of the Merton implied Sharpe ratio arose.

and comparing with the expansion

$$V^a(t,x,y,w) = M(t,w;\lambda_0) + aV^{(1)}(t,x,y,w) + \cdots$$

yields $\Lambda^{(0)} = \lambda_0$ and

$$\Lambda^{(1)} = \frac{V^{(1)}(t,x,y,w)}{M_\lambda(t,w;\lambda^{(0)})}. \tag{46}$$

Next, from Lemma 2.1, we have

$$M_\lambda(t,w;\lambda^{(0)}) = -(T-t)\lambda_0\mathcal{D}_2 M(t,w;\lambda^{(0)}) = -(T-t)\lambda_0\mathcal{D}_2 V^{(0)}(t,w) = (T-t)\lambda_0\mathcal{D}_1 V^{(0)}(t,w).$$

Then, using the formula (44) for $V^{(1)}$ in (46) gives

$$\Lambda^{(1)}(t,x,y,w) = A(t,x,y) + \tfrac{1}{2}(T-t)B\,\frac{\mathcal{D}_1^2 V^{(0)}(t,w)}{\mathcal{D}_1 V^{(0)}(t,w)}.$$

By computing

$$\frac{\mathcal{D}_1^2 V^{(0)}(t,w)}{\mathcal{D}_1 V^{(0)}(t,w)} = R_w(t,w;\lambda_0) - 1,$$

we have

$$\Lambda^a \approx \Lambda^{(0)} + a\Lambda^{(1)} = \lambda_0 + a\left[A(t,x,y) + \tfrac{1}{2}(T-t)B\left(R_w(t,w;\lambda_0) - 1\right)\right], \tag{47}$$

where R is the Merton risk tolerance function defined in (10).

3.6 Optimal Portfolio

From (6), we have that the optimal strategy $\pi^{a,*}$ is given by

$$\pi^{a,*} = -\frac{\mu^a(x,y)V_w^a}{(\sigma^a)^2(x,y)V_{ww}^a} - \frac{\rho\beta^a(x,y)V_{yw}^a}{\sigma^a(x,y)V_{ww}^a} - \frac{V_{xw}^a}{V_{ww}^a}. \tag{48}$$

It is convenient in deriving a compact form for our portfolio approximation to write our first order approximation to the value function as the Merton value function evaluated at the first order series (47) for the Sharpe ratio

$$V^a(t,x,y,w) \approx \bar{V}(t,x,t,w) := M(t,w;\lambda_0 + a\Lambda^{(1)}(t,x,y,w)).$$

Then our approximate first order policy will be to substitute $\bar{V}$ for V^a in (48).

We have

$$\begin{aligned}
\bar{V}_w(t,x,y,w) &= M_w\left(t,w;\lambda_0 + a\Lambda^{(1)}(t,x,y,w)\right) + aM_\lambda(t,w;\lambda_0)\tfrac{1}{2}(T-t)BR_{ww}(t,w;\lambda_0) + \mathcal{O}(a^2),\\
\bar{V}_{ww}(t,x,y,w) &= M_{ww}\left(t,w;\lambda_0 + a\Lambda^{(1)}(t,x,y,w)\right) + a\tfrac{1}{2}(T-t)B\left(R_{ww}(t,w;\lambda_0)M_\lambda(t,w;\lambda_0)\right)_w + \mathcal{O}(a^2),\\
\bar{V}_{yw}(t,x,y,w) &= a\lambda_{0,1}M_{\lambda w}(t,w;\lambda_0) + \mathcal{O}(a^2),\\
\bar{V}_{xw}(t,x,y,w) &= a\lambda_{1,0}M_{\lambda w}(t,w;\lambda_0) + \mathcal{O}(a^2),
\end{aligned}$$

where $\mathcal{O}(a^2)$ denotes series terms in powers of a^2 and higher.

Let us compute

$$-\frac{\bar{V}_w}{\bar{V}_{ww}} = -\frac{M_w\left(t,w;\lambda_0 + a\Lambda^{(1)}(t,x,y,w)\right) + aM_\lambda(t,w;\lambda_0)\tfrac{1}{2}(T-t)BR_{ww}(t,w;\lambda_0)}{M_{ww}\left(t,w;\lambda_0 + a\Lambda^{(1)}(t,x,y,w)\right) + a\tfrac{1}{2}(T-t)B\left(R_{ww}(t,w;\lambda_0)M_\lambda(t,w;\lambda_0)\right)_w}$$

$$
\begin{aligned}
&= R\left(t,w;\lambda_0 + a\Lambda^{(1)}(t,x,y,w)\right) - a\frac{M_\lambda}{M_{ww}}\tfrac{1}{2}(T-t)BR_{ww} + a\tfrac{1}{2}(T-t)B\frac{M_w}{M_{ww}^2}(R_{ww}M_\lambda)_w + \mathcal{O}(a^2). \\
&= R\left(t,w;\lambda_0 + a\Lambda^{(1)}(t,x,y,w)\right) + a\tfrac{1}{2}(T-t)^2B\lambda_0R^2(R_{ww} + RR_{www} + (R_w-1)R_{ww}) + \mathcal{O}(a^2).
\end{aligned}
$$

Here we have used the following identities satisfied by the Merton value function $M(t,w;\lambda)$ and its risk tolerance function $R(t,w;\lambda)$:

$$
\frac{M_\lambda}{M_{ww}} = -(T-t)\lambda R^2, \tag{49}
$$

$$
\frac{M_w}{M_{ww}^2}M_\lambda = (T-t)\lambda R^3, \tag{50}
$$

$$
\frac{M_w}{M_{ww}^2}M_{\lambda w} = (T-t)\lambda R^2(R_w-1), \tag{51}
$$

where (49) comes from Lemma 2.1; (50) comes from multiplying (49) by $-R$; and in the last expression (51), we also use $R^2M_{www} = (R_w+1)M_w$.

Additionally, we compute

$$
\begin{aligned}
-\frac{\bar V_{yw}}{\bar V_{ww}} &= -a\lambda_{0,1}\frac{M_{\lambda w}}{M_{ww}}(t,w;\lambda_0) = a\lambda_{0,1}(T-t)\lambda_0R(t,w;\lambda_0)\left(R_w(t,w;\lambda_0)-1\right) + \mathcal{O}(a^2), \\
-\frac{\bar V_{xw}}{\bar V_{ww}} &= -a\lambda_{1,0}\frac{M_{\lambda w}}{M_{ww}}(t,w;\lambda_0) = a\lambda_{1,0}(T-t)\lambda_0R(t,w;\lambda_0)\left(R_w(t,w;\lambda_0)-1\right) + \mathcal{O}(a^2).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\pi^{a,*} \approx{}& \frac{\mu^a(x,y)}{(\sigma^a)^2(x,y)}\left\{R\left(t,w;\lambda_0 + a\Lambda^{(1)}(t,x,y,w)\right) + \tfrac{1}{2}a(T-t)^2B\lambda_0R^2(RR_{www} + (R+R_w-1)R_{ww})\right\} \\
&+ a(T-t)\lambda_0R(R_w-1)\left(\frac{\rho\beta^a(x,y)}{\sigma^a(x,y)}\lambda_{0,1} + \lambda_{1,0}\right),
\end{aligned}
$$

where R without an argument denotes $R(t,w;\lambda_0)$. One could substitute the first two terms of the polynomial expansion of the coefficients, but since they are assumed known, there is no loss in accuracy in using the full expressions. The first order approximate optimal strategy is written in terms of the risk tolerance function and its derivatives.

3.7 Summary

We collect here the expressions for our first order approximation formulas, which follow from the prior calculations and setting the accounting parameter $a = 1$.

- Our first order approximation to the value function $V(t,x,y,w)$ in (3), solution of the PDE problem (7) is given by $V(t,x,y,w) \approx \bar V(t,x,y,w)$, where

$$
\begin{aligned}
\bar V(t,x,y,w) &= V^{(0)}(t,w) + V^{(1)}(t,x,y,w) \\
&= M(t,w;\lambda_0) + \left((T-t)\lambda_0A(t,x,y)\,\mathcal{D}_1 + \tfrac{1}{2}(T-t)^2\lambda_0B\,\mathcal{D}_1^2\right)M(t,w;\lambda_0),
\end{aligned}
$$

  and A and B are given in (43).

- The implied Sharpe ratio $\Lambda = \Lambda(t,x,y,w)$ defined by $V(t,x,y,w) = M(t,w;\Lambda)$ is approximated to first order by $\Lambda \approx \bar\Lambda$, where

$$
\begin{aligned}
\bar\Lambda(t,x,y,w) &= \Lambda^{(0)} + \Lambda^{(1)} \\
&= \lambda_0 + A(t,x,y) + \tfrac{1}{2}(T-t)B\left(R_w(t,w;\lambda_0)-1\right).
\end{aligned} \tag{52}
$$

- Our first order approximation to the optimal strategy $\pi^*(t,x,y,w)$ in (6) is given by $\pi^* \approx \bar{\pi}$, where

$$\bar{\pi}(t,x,y,w) = \frac{\mu(x,y)}{\sigma^2(x,y)} \Big\{ R\left(t,w;\lambda_0 + \Lambda^{(1)}(t,x,y,w)\right) + \tfrac{1}{2}(T-t)^2 B\lambda_0 R^2 (RR_{www} + (R+R_w-1)R_{ww}) \Big\} + (T-t)\lambda_0 R(R_w - 1)\left(\frac{\rho\beta(x,y)}{\sigma(x,y)}\lambda_{0,1} + \lambda_{1,0}\right). \tag{53}$$

This formula has principle term that is the classical Merton strategy $-\frac{\mu}{\sigma^2}R$, but here is updated to account for LSV by using the current $\mu(x,y)$ and $\sigma(x,y)$ values, and with the implied Sharpe ratio in the risk tolerance function. The other terms contain effects of correlation ρ, volatility of volatility β, higher Taylor expansion terms of the stochastic Sharpe ratio, and higher derivatives of the the risk tolerance function with respect to wealth. Even for a utility function where there is no explicit solution for the constant parameter Merton value function M, the risk tolerance is easily computed by numerically solving Black's equation, as detailed in (Fouque et al., 2013, Section 6.2).

4 Higher Order Terms

Having obtained PDEs for $V^{(0)}$ and $V^{(1)}$, we examine the higher order terms. An exercise in accounting shows that for all $n \geq 1$ the function $V^{(n)}(t,x,y,w)$ satisfies a linear PDE of the form

$$\left(\frac{\partial}{\partial t} + \mathcal{A}_0 + \mathcal{B}_0\right)V^{(n)} + H_n = 0, \qquad V^{(n)}(T,x,y,w) = 0, \tag{54}$$

where the source term H_n depends only on $V^{(k)}$ $(k \leq n-1)$. To see this, observe that the nth-order PDE involves three types of terms

$$\mathcal{O}(a^n): \qquad V_t^{(n)}, \qquad \mathcal{A}_k V^{(n-k)}, \quad (k \leq n), \qquad \sum_{j+k+l+m=n} \chi_j \, V_{\alpha w}^{(k)} V_{\gamma w}^{(l)} \left(\frac{1}{V_{ww}^a}\right)_m, \tag{55}$$

where, in the last term, χ is a place holder for one of the coefficient functions appearing in $\mathcal{N}^a$, the symbols (α, δ) are place holders for (x,y) or null (meaning just a single derivative in w), and $\left(\frac{1}{V_{ww}^a}\right)_m$ is the mth order term in the Taylor series expansion of $\left(\frac{1}{V_{ww}^a}\right)$ about the point $a=0$, i.e.,

$$\left(\frac{1}{V_{ww}^a}\right) = \frac{1}{V_{ww}^{(0)}} + \sum_{k=1}^{\infty} a^k \left(\frac{1}{V_{ww}^a}\right)_k, \qquad \left(\frac{1}{V_{ww}^a}\right)_k = \sum_{m=1}^{k} \frac{(-1)^m}{(V_{ww}^{(0)})^{1+m}} \left(\sum_{i \in I_{k,m}} \prod_{j=1}^{m} V_{ww}^{(i_j)}\right), \tag{56}$$

where $I_{k,m}$ is given by

$$I_{k,m} = \left\{ i = (i_1, i_2, \cdots, i_m) \in \mathbb{N}^m : \sum_{j=1}^{m} i_j = k \right\}. \tag{57}$$

The terms in (54) that involve $V^{(n)}$ are precisely those terms that appear in $(\frac{\partial}{\partial t} + \mathcal{A}_0 + \mathcal{B}_0)V^{(n)}$. The terms that do not involve $V^{(n)}$ are grouped into the source term H_n. We provide here an explicit expression for the second order source term H_2, which appears in the $\mathcal{O}(a^2)$ PDE:

$$H_2 = -(\tfrac{1}{2}\lambda^2)_2 \frac{(V_w^{(0)})^2}{V_{ww}^{(0)}} - (H_1 + \mathcal{B}_0 V^{(1)}) \cdot \frac{V_{ww}^{(1)}}{V_{ww}^{(0)}} + \mathcal{A}_1 V^{(1)} - (\tfrac{1}{2}\lambda^2)_0 \frac{(V_w^{(1)})^2}{V_{ww}^{(0)}}$$

$$
\begin{aligned}
&- 2(\tfrac{1}{2}\lambda^2)_1 \frac{(V_w^{(0)})(V_w^{(1)})}{V_{ww}^{(0)}} - (\rho\beta\lambda)_0 \frac{(V_w^{(1)})(V_{yw}^{(1)})}{V_{ww}^{(0)}} - (\rho\beta\lambda)_1 \frac{(V_w^{(0)})(V_{yw}^{(1)})}{V_{ww}^{(0)}} \\
&- (\tfrac{1}{2}\rho^2\beta^2)_0 \frac{(V_{yw}^{(1)})^2}{V_{ww}^{(0)}} - \mu_0 \frac{(V_w^{(1)})(V_{xw}^{(1)})}{V_{ww}^{(0)}} - \mu_1 \frac{(V_w^{(0)})(V_{xw}^{(1)})}{V_{ww}^{(0)}} \\
&- (\rho\sigma\beta)_0 \frac{(V_{xw}^{(1)})(V_{yw}^{(1)})}{V_{ww}^{(0)}} - (\tfrac{1}{2}\sigma^2)_0 \frac{(V_{xw}^{(1)})^2}{V_{ww}^{(0)}}.
\end{aligned}
$$

Higher-order sources terms can be obtained systematically using a computer algebra program such as Wolfram Mathematica.

Now let $q^{(n)}$ be defined from $V^{(n)}$ by

$$V^{(n)}(t,x,y,w) = q^{(n)}(t,x,y,z(t,w)),$$

using the transformation (24). Then, using Lemma 3.3, we see that the PDE (54) for $V^{(n)}$, which has (t,w)-dependent coefficients through the dependence of $\mathcal{B}_0$ in (20) on $R(t,w;\lambda_0)$, is transformed to the *constant coefficient* equation for $q^{(n)}$:

$$\left(\frac{\partial}{\partial t} + \mathcal{A}_0 + \mathcal{C}_0\right) q^{(n)} + Q_n = 0, \qquad q^{(n)}(T,x,y,z) = 0. \tag{58}$$

The source term is found from $H_n(t,x,y,w) = Q_n(t,x,y,z(t,w))$.

We must establish that, for every $n \geq 1$ there exists a function Q_n such that $Q_n(t,x,y,z(t,w)) = H_n(t,x,y,w)$. From (55) we see that the source term H_n contains two types of terms, the first of which is $\mathcal{A}_k V^{(n-k)}$ $(1 \leq k \leq n)$. Since $\mathcal{A}_k$ acts only on (x,y), we have that $\mathcal{A}_k V^{(n-k)} = \mathcal{A}_k q^{(n-k)}$. The second sort of term appearing in (55) are those of the form

$$\sum_{j+k+l+m=n} \chi_j \frac{V_{\alpha w}^{(k)} V_{\gamma w}^{(l)}}{V_{ww}^{(0)}} \sum_{p=1}^{m} (-1)^p \left(\sum_{i \in I_{k,p}} \prod_{j=1}^{p} \frac{V_{ww}^{(i_j)}}{V_{ww}^{(0)}} \right), \qquad k,l,m \leq n-1, \tag{59}$$

where we have used (56).

Next, using

$$\frac{V_{\alpha w}^{(k)} V_{\gamma w}^{(l)}}{V_{ww}^{(0)}} = -\frac{q_{\alpha z}^{(k)} q_{\gamma z}^{(l)}}{q_z^{(0)}}, \qquad \frac{V_{ww}^{(i)}}{V_{ww}^{(0)}} = \frac{-q_{zz}^{(i)}}{q_z^{(0)}} + \frac{(q_z^{(0)} + q_{zz}^{(0)})q_z^{(i)}}{(q_z^{(0)})^2},$$

we see that (59) can be written as

$$\sum_{j+k+l+m=n} \chi_j \frac{-q_{\alpha z}^{(k)} q_{\gamma z}^{(l)}}{q_z^{(0)}} \sum_{p=1}^{m} (-1)^p \left(\sum_{i \in I_{k,p}} \prod_{j=1}^{p} \left(\frac{-q_{zz}^{(i_j)}}{q_z^{(0)}} + \frac{(q_z^{(0)} + q_{zz}^{(0)})q_z^{(i_j)}}{(q_z^{(0)})^2} \right) \right), \tag{60}$$

where $(l,k,m \leq n-1)$. We have therefore established that, for every $n \geq 1$, the source term $H_n(t,x,y,w)$, which is composed of products and quotients of derivatives of $V^{(k)}(t,x,y,w)$ $(k \leq n-1)$, can be written be written as a function Q_n, which is composed of products and quotients of derivatives of $q^{(k)}(t,x,y,z)$ $(k \leq n-1)$.

In Proposition 3.6, we saw that $q^{(1)}$, the first-order transformed value function, can be expressed as a differential operator acting on $q^{(0)}$, specifically $q^{(1)} = \mathcal{L}_1 q^{(0)}$, where

$$\mathcal{L}_1 = \left[(T-t)(\tfrac{1}{2}\lambda^2)_1(x,y)I + \tfrac{1}{2}(T-t)^2 \left((\tfrac{1}{2}\lambda^2)_{1,0}\mathcal{L}_X + (\tfrac{1}{2}\lambda^2)_{0,1}\mathcal{L}_Y\right)\right] \frac{\partial}{\partial z}. \tag{61}$$

We will show that, for certain utility functions U, each of the higher order terms $q^{(n)}$ $(n \geq 2)$ can also be written as a differential operator acting on $q^{(0)}$. From Proposition 3.5, we know that if the source Q_n in the nth order PDE (58) is of the form (34), then this will be the case. From (32), we see that

$$Q_1 = \mathcal{Q}_1 q_0 \qquad \text{where} \qquad \mathcal{Q}_1 = (\tfrac{1}{2}\lambda^2)_1(x,y)\frac{\partial}{\partial z}, \tag{62}$$

Unfortunately, this is not always the case.

To see this, we examine Q_2, the source term in the PDE for $q^{(2)}$, which one can compute:

$$\begin{aligned} Q_2 = (\tfrac{1}{2}\lambda^2)_2 q_z^{(0)} &- \left(\frac{(q_z^{(0)} + q_{zz}^{(0)})q_z^{(1)}}{(q_z^{(0)})^2} - \frac{q_{zz}^{(1)}}{q_z^{(0)}}\right)(Q_1 + \mathcal{C}_0 q_1) + \mathcal{A}_1 q_1 \\ &+ (\tfrac{1}{2}\lambda^2)_0 \frac{(q_z^{(1)})^2}{q_z^{(0)}} + 2(\tfrac{1}{2}\lambda^2)_1 q_z^{(1)} + (\rho\beta\lambda)_0 \frac{q_z^{(1)} q_{yz}^{(1)}}{q_z^{(0)}} \\ &+ (\rho\beta\lambda)_1 q_{yz}^{(1)} + (\tfrac{1}{2}\rho^2\beta^2)_0 \frac{q_{yz}^{(1)} q_{yz}^{(1)}}{q_z^{(0)}} + \mu_0 \frac{q_z^{(1)} q_{xz}^{(1)}}{q_z^{(0)}} \\ &+ \mu_1 q_{xz}^{(1)} + (\rho\sigma\beta)_0 \frac{q_{xz}^{(1)} q_{yz}^{(1)}}{q_z^{(0)}} + (\tfrac{1}{2}\sigma^2)_0 \frac{q_{xz}^{(1)} q_{xz}^{(1)}}{q_z^{(0)}}. \end{aligned} \tag{63}$$

From (63) we see that Q_2 can be written as $Q_2 = \mathcal{Q}_2 q_0$ where

$$\begin{aligned} \mathcal{Q}_2 = (\tfrac{1}{2}\lambda^2)_2 \frac{\partial}{\partial z} &- \left(\frac{(q_z^{(0)} + q_{zz}^{(0)})q_z^{(1)}}{(q_z^{(0)})^2} - \frac{q_{zz}^{(1)}}{q_z^{(0)}}\right)(\mathcal{Q}_1 + \mathcal{C}_0 \mathcal{L}_1) + \mathcal{A}_1 \mathcal{L}_1 \\ &+ (\tfrac{1}{2}\lambda^2)_0 \frac{(q_z^{(1)})}{q_z^{(0)}} \frac{\partial}{\partial z}\mathcal{L}_1 + 2(\tfrac{1}{2}\lambda^2)_1 \frac{\partial}{\partial z}\mathcal{L}_1 + (\rho\beta\lambda)_0 \frac{(q_{yz}^{(1)})}{q_z^{(0)}} \frac{\partial}{\partial z}\mathcal{L}_1 \\ &+ (\rho\beta\lambda)_1 \frac{\partial^2}{\partial y \partial z}\mathcal{L}_1 + (\tfrac{1}{2}\rho^2\beta^2)_0 \frac{(q_{yz}^{(1)})}{q_z^{(0)}} \frac{\partial^2}{\partial y \partial z}\mathcal{L}_1 + \mu_0 \frac{(q_{xz}^{(1)})}{q_z^{(0)}} \partial_z \mathcal{L}_1 \\ &+ \mu_1 \frac{\partial^2}{\partial x \partial z}\mathcal{L}_1 + (\rho\sigma\beta)_0 \frac{(q_{xz}^{(1)})}{q_z^{(0)}} \frac{\partial^2}{\partial y \partial z}\mathcal{L}_1 + (\tfrac{1}{2}\sigma^2)_0 \frac{(q_{xz}^{(1)})}{q_z^{(0)}} \frac{\partial^2}{\partial x \partial z}\mathcal{L}_1, \end{aligned} \tag{64}$$

where $\mathcal{Q}_1$ was given in (62), and $\mathcal{L}_1$ in (61).

In order to use Proposition 3.5, we must establish that coefficients of $\mathcal{Q}_2$ are polynomials in (x, y, z). The complicating terms are those that contain derivatives of $q^{(0)}$ and $q^{(1)}$ divided by $q_z^{(0)}$. Such terms are always polynomials in (x, y), but may not be polynomial in z. The following lemma provides conditions under which the differential operator $\mathcal{Q}_n$ is guaranteed to have coefficients that are independent of z:

Lemma 4.1. *Suppose $q^{(0)}(t, z)$ is of the form:*

$$q^{(0)}(t,z) = a(t)\mathrm{e}^{b(t)+zc(t)}. \tag{65}$$

Then, for every $n \geq 1$, the source term Q_n appearing in PDE (58) can be written as $Q_n = \mathcal{Q}_n q^{(0)}$, where the differential operator $\mathcal{Q}_n$ has coefficients that are polynomial in (x, y) and independent of z.

Proof. We will prove by induction on n that there exists a differential operator $\mathcal{Q}_n$ whose coefficients are polynomial in (x, y), independent of z, and which satisfies $Q_n = \mathcal{Q}_n q^{(0)}$, where Q_n is the nth-order source term appearing in (58). We know from (62) that such a $\mathcal{Q}_1$ exists. We now assume such $\mathcal{Q}_k$ exist for all $1 \leq k \leq n-1$, and we show that $\mathcal{Q}_n$ exists and has the required form.

The existence $\mathcal{Q}_k$ implies from Proposition 3.5 that there exists an operator $\mathcal{L}_k$ such that $q^{(k)} = \mathcal{L}_k q^{(0)}$. Moreover, since $\mathcal{Q}_k$ is polynomial in (x, y) and independent of z it follows from (41) that $\mathcal{L}_k$ has coefficients that are polynomial in (x, y) and independent of z. Now, we recall that Q_n contains two types of terms: $\mathcal{A}_k q^{(n-k)}$ $(1 \leq k \leq n)$ and terms of the form (60). Let us first examine terms of the form $\mathcal{A}_k q^{(n-k)}$ $(1 \leq k \leq n)$. Note that

$$\mathcal{A}_k q^{(n-k)} = \mathcal{A}_k \mathcal{L}_{n-k} q^{(0)}, \qquad 1 \leq k \leq n.$$

The coefficients of $\mathcal{A}_k$ are polynomial in (x, y) and independent of z by construction. Hence, the coefficients of $\mathcal{A}_k \mathcal{L}_{n-k}$ are also polynomial in (x, y) and independent of z. Now, let us examine the terms of the form (60). Using $q^{(k)} = \mathcal{L}_k q^{(0)}$ we can express (60) as

$$\sum_{j+k+l+m=n} \chi_j \frac{-(\frac{\partial^2}{\partial \alpha \partial z} \mathcal{L}_k q^{(0)})\,(\frac{\partial^2}{\partial \gamma \partial z} \mathcal{L}_l q^{(0)})}{q_z^{(0)}} \sum_{p=1}^{m} (-1)^p \left(\sum_{i \in I_{k,p}} \prod_{j=1}^{p} \left(\frac{(q_z^{(0)} + q_{zz}^{(0)})(\partial_z \mathcal{L}_{i_j} q^{(0)})}{(q_z^{(0)})^2} - \frac{\frac{\partial^2}{\partial z^2} \mathcal{L}_{i_j} q^{(0)}}{q_z^{(0)}} \right) \right), \tag{66}$$

where $l, k, m \leq n-1$. Since, by assumption, $q^{(0)}$ is of the form (65), it follows that terms of the form (66) are polynomials in (x, y) and independent of z. We have therefore established that Q_n can be written as $\mathcal{Q}_n q^{(0)}$ where $\mathcal{Q}_n$ is a differential operator whose coefficients are polynomial in (x, y) and independent of z. □

We will see in the next section that, when U belongs to the power utility class, then $q^{(0)}$ is of the form (65). Thus, the nth-order term $q^{(n)}$ can be written as a differential operator $\mathcal{L}_n$ acting on $q^{(0)}$.

5 Specific results for power utility

In this section, we consider the case where the utility function U belongs to the power utility class

$$\text{Power utility:} \qquad U(w) = \frac{w^{1-\gamma}}{1-\gamma}, \qquad \gamma > 0,\ \gamma \neq 1. \tag{67}$$

For general LSV dynamics (2), we will obtain the second-order approximation for the value function u, optimal investment strategy π^* and implied Sharpe ratio Λ. Then, we will establish error estimates for the approximate value function in a stochastic volatility setting.

5.1 Value function

To obtain the second order approximation to the value function V, we must first compute $q^{(0)}$, $q^{(1)}$ and $q^{(2)}$. With U given by (67), we have $U'(w) = w^{-\gamma}$ and $[U']^{-1}(\zeta) = \zeta^{-1/\gamma}$. As found in Merton (1969), we have

$$V^{(0)}(t, w) = M(t, w; \lambda_0) = \frac{w^{1-\gamma}}{1-\gamma} \exp\left(\tfrac{1}{2} \left(\frac{1-\gamma}{\gamma} \right) \lambda_0^2 (T-t) \right). \tag{68}$$

Then the transform variable z in (24) is given by

$$z(t,w) = \gamma \log w + \tfrac{1}{2}\left(\tfrac{2\gamma-1}{\gamma}\right)\lambda_0^2(T-t), \tag{69}$$

and the solution of the heat equation PDE problem (28) is

$$q^{(0)}(t,z) = \frac{1}{1-\gamma}\exp\left(\frac{1-\gamma}{\gamma}\left(z+\tfrac{1}{2}\left(\tfrac{1-\gamma}{\gamma}\right)\lambda_0^2(T-t)\right)\right). \tag{70}$$

Next, using (42) and (70), we compute $q^{(1)}$:

$$\begin{aligned} q^{(1)}(t,x,y,z) = \frac{1-\gamma}{\gamma}\Bigg(& \left(\tfrac{1}{2}\lambda^2\right)_{1,0}\left((T-t)(x-\bar{x})+\tfrac{1}{2}(T-t)^2\left(\tfrac{1}{\gamma}\mu_0-\tfrac{1}{2}\sigma_0^2\right)\right) \\ & +\left(\tfrac{1}{2}\lambda^2\right)_{0,1}\left((T-t)(y-\bar{y})+\tfrac{1}{2}(T-t)^2\left(c_0+\tfrac{1-\gamma}{\gamma}\rho\beta_0\lambda_0\right)\right)\Bigg)q^{(0)}(t,z), \end{aligned}$$

from which we obtain

$$\begin{aligned} V^{(1)}(t,x,y,z) = \frac{1-\gamma}{\gamma}\Bigg(& \left(\tfrac{1}{2}\lambda^2\right)_{1,0}\left((T-t)(x-\bar{x})+\tfrac{1}{2}(T-t)^2\left(\tfrac{1}{\gamma}\mu_0-\tfrac{1}{2}\sigma_0^2\right)\right) \\ & +\left(\tfrac{1}{2}\lambda^2\right)_{0,1}\left((T-t)(y-\bar{y})+\tfrac{1}{2}(T-t)^2\left(c_0+\tfrac{1-\gamma}{\gamma}\rho\beta_0\lambda_0\right)\right)\Bigg)V^{(0)}(t,z), \end{aligned} \tag{71}$$

where, as a reminder, $(\frac{1}{2}\lambda^2)_{1,0}$ and $(\frac{1}{2}\lambda^2)_{0,1}$ are given by

$$(\tfrac{1}{2}\lambda^2)_{1,0} = \left(\frac{\mu^2}{2\sigma^2}\right)_x(\bar{x},\bar{y}), \qquad (\tfrac{1}{2}\lambda^2)_{0,1} = \left(\frac{\mu^2}{2\sigma^2}\right)_y(\bar{x},\bar{y}).$$

Having obtained an explicit expressions for $q^{(0)}$ and $q^{(1)}$, we can now compute $q^{(2)}$. The second order source term Q_2, given by (63), can be written as $Q_2 = \mathcal{Q}_2 q_0$ where the operator $\mathcal{Q}_2$ is given by (64). From (70), we see that $q^{(0)}$ is of the form (65). Thus, from Lemma 4.1 we know that the coefficients of $\mathcal{Q}_2$ are polynomial in (x,y) and independent of z. Therefore, we can use Proposition 3.5 to compute $q^{(2)} = \mathcal{L}_2 q_0$. The expression for $q^{(2)}$ is quite long. As such, for the sake of brevity, we do not include it here.

We can obtain $V^{(2)}$ from $q^{(2)}$ using (30) and (69). The same procedure can be used to compute higher-order terms: $q^{(n)}$ $(n \geq 3)$. Since the expressions for $u^{(2)}$ and higher-order terms are quite long, we do not present them here. However, in the numerical examples that follow, we do compute the second order approximation, and we will see that it provides a noticeably more accurate approximation of V than does the first order approximation.

5.2 Optimal Strategy

For power utility, the Merton risk tolerance function is especially simple: $R(t,w;\lambda) = w/\gamma$, and it does not depend on t, T or the Sharpe ratio λ. Therefore, the approximate first order optimal strategy in (53) is given by $\pi^* \approx \bar{\pi}$, where

$$\bar{\pi}(t,x,y,w) = \left[\frac{\mu(x,y)}{\sigma^2(x,y)} + (T-t)\lambda_0\left(\frac{1}{\gamma}-1\right)\left(\frac{\rho\beta(x,y)}{\sigma(x,y)}\lambda_{0,1}+\lambda_{1,0}\right)\right]\frac{w}{\gamma}, \tag{72}$$

which is also proportional to the current wealth level w as in the classical Merton strategy, but with proportion that varies with the model coefficients whose values move with the log stock price x and the volatility driving factor y.

We remark that it is also possible to compute the next order of the strategy approximation π_2^* in the case of power utility using the lengthy expression for $V^{(2)}$. For the special case $(x,y) = (\bar{x},\bar{y})$, we have

$$\pi_2^* = w \times \Bigg[\frac{(T-t)^2(\gamma-1)}{2\gamma^3} \Big((\gamma-1)(\tfrac{1}{2}\lambda^2)_{0,1}(\rho\beta\lambda)_{1,0} + \gamma(\tfrac{1}{2}\lambda^2)_{1,0}(\tfrac{1}{2}\sigma^2)_{1,0} - \left(\gamma c_{1,0}(\tfrac{1}{2}\lambda^2)_{0,1} + (\gamma c_0 - (\gamma-1)(\rho\beta\lambda)_0)\,(\tfrac{1}{2}\lambda^2)_{1,1} + 2\mu_0(\tfrac{1}{2}\lambda^2)_{2,0} - \gamma\sigma_0^2(\tfrac{1}{2}\lambda^2)_{2,0} + (\tfrac{1}{2}\lambda^2)_{1,0}\mu_{1,0}\right) \Big)$$

$$+ \frac{(T-t)^3(\gamma-1)^2}{8\gamma^4(\frac{1}{2}\sigma^2)_0} \Big((\rho\sigma\beta)_0(\tfrac{1}{2}\lambda^2)_{0,1} \left(-2\left(\gamma c_0 - (\gamma-1)(\rho\beta\lambda)_0\right)(\tfrac{1}{2}\lambda^2)_{0,1} + \left(-2\mu_0 + \gamma\sigma_0^2\right)(\tfrac{1}{2}\lambda^2)_{1,0}\right) \Big) \Bigg].$$

5.3 Implied Sharpe ratio

We now compute the first order approximation of the implied Sharpe ratio Λ, which was introduced in Section 3.5. From (52), we have $\Lambda \approx \bar{\Lambda}$, where

$$\bar{\Lambda} = \lambda_0 + \lambda_{1,0}(x-\bar{x}) + \lambda_{0,1}(y-\bar{y}) + \frac{1}{2}(T-t)\Big(\lambda_{0,1}\left(c_0 + \tfrac{1-\gamma}{\gamma}(\rho\beta\lambda)_0\right) + \lambda_{1,0}\left(\tfrac{1}{\gamma}\mu_0 - (\tfrac{1}{2}\sigma_0^2)\right)\Big). \quad (73)$$

The second order correction Λ_2 is quite long, and we omit it for the sake of brevity.

Observe that, for power utility, in which an explicit expression for the constant parameter Merton value function M is available, one can obtain an expression for the implied Sharpe ratio Λ by solving (45) with M given by (68):

$$\Lambda = \sqrt{\log\left(\frac{V}{U(w)}\right)\frac{2\gamma}{(1-\gamma)(T-t)}}. \quad (74)$$

This will be useful when we test the numerical accuracy of the Sharpe ratio approximation in two examples.

5.4 Accuracy of the approximation for stochastic volatility models

In this section, we establish the accuracy of

$$\bar{V}^{(n)} = \sum_{k=0}^{n} V^{(k)},$$

the nth-order approximation of the value function V, assuming stochastic volatility dynamics of the form

$$\begin{aligned} \mathrm{d}X_t &= \left(\mu(Y_t) - \frac{1}{2}\sigma^2(Y_t)\right)\mathrm{d}t + \sigma(Y_t)\mathrm{d}B_t^X, \\ \mathrm{d}Y_t &= c(Y_t)\mathrm{d}t + \beta(Y_t)\mathrm{d}B_t^Y, \\ \mathrm{d}\langle B^X, B^Y\rangle_t &= \rho\,\mathrm{d}t, \end{aligned} \quad (75)$$

and a utility function U of the power utility class (67).

Throughout this section, we will make the following assumption:

Assumption 5.1. There exists a constant $C > 0$ such that the following holds:
(i) *Uniform ellipticity*: $1/C \le \beta^2 \le C$.
(ii) *Regularity and boundedness*: The coefficients c, $\rho\beta\lambda$, β^2 and λ^2 are $C^{n+1}(\mathbb{R})$ and all derivatives up to order n are bounded by C.
(iii) The *risk aversion parameter* in the utility function (67) satisfies $\gamma > 1$.

Clearly, stochastic volatility dynamics (75) are a special case of the more general local-stochastic volatility dynamics (2). As such, one can obtain a series approximation $\bar{V}^{(n)}$ of the value function $V^a(t, y, w)$, which is in this case independent of x, by solving the sequence of PDEs (54). An alternative but equivalent approach is to linearize the full PDE (13), and then perform a series approximation on the resulting linear PDE. This is the approach we follow here.

Assuming power utility (67) and dynamics given by (75), Zariphopoulou (2001) shows that the function $V^a(t, y, w)$, solution of (13), is given by

$$V^a(t, w, y) = \frac{w^{1-\gamma}}{1-\gamma} (\psi^a(t, y))^\eta, \qquad \eta = \frac{\gamma}{\gamma + (1-\gamma)\rho^2}, \tag{76}$$

where the function ψ^a satisfies the Cauchy problem

$$0 = (\partial_t + \widehat{\mathcal{A}}^a)\psi^a, \qquad \psi^a(T, y) = 1, \qquad a \in [0, 1], \tag{77}$$

and $\widehat{\mathcal{A}}^a$ is a linear elliptic operator given by

$$\widehat{\mathcal{A}}^a = (\tfrac{1}{2}\beta^2)^a \partial_y^2 + \left(c^a + \tfrac{1-\gamma}{\gamma}(\rho\beta\lambda)^a\right)\partial_y + \tfrac{1-\gamma}{\eta\gamma}(\tfrac{1}{2}\lambda^2)^a. \tag{78}$$

Let us denote $\widehat{\mathcal{A}} = \widehat{\mathcal{A}}^a|_{a=1}$ and $\psi = \psi^a|_{a=1}$.

Remark 5.2. Assumption 5.1 part (iii) guarantees that the last term in (78) is strictly negative.

Remark 5.3. The linearization transformation described above works only for one-factor pure stochastic volatility dynamics (75), or complete market pure local volatility models, and only for power utility (67). For more general local-stochastic volatility dynamics (2) and utility functions U, one must work with nonlinear PDE (7).

We return now to (77). Noting that $\widehat{\mathcal{A}}^a$ can be written as

$$\widehat{\mathcal{A}}^a = \sum_{n=0}^{\infty} a^n \widehat{\mathcal{A}}_n, \qquad \widehat{\mathcal{A}}_n = (\tfrac{1}{2}\beta^2)_n \partial_y^2 + \left(c_n + \frac{1-\gamma}{\gamma}(\rho\beta\lambda)_n\right)\partial_y + \frac{1-\gamma}{\eta\gamma}(\tfrac{1}{2}\lambda^2)_n, \tag{79}$$

we seek a solution ψ^a to (77) of the form

$$\psi^a = \sum_{n=0}^{\infty} a^n \psi_n. \tag{80}$$

Inserting (79) and (80) into PDE (77) and collecting terms of like powers of a we obtain the following sequence of nested PDEs:

$$\mathcal{O}(1): \qquad 0 = (\partial_t + \widehat{\mathcal{A}}_0)\psi_0, \qquad \psi_0(T, y) = 1, \tag{81}$$

$$\mathcal{O}(a^n): \qquad 0 = (\partial_t + \widehat{\mathcal{A}}_0)\psi_n + \sum_{k=1}^{n} \widehat{\mathcal{A}}_k \psi_{n-k}, \qquad \psi_n(T, y) = 0. \tag{82}$$

This sequence of nested PDEs has been solved explicitly in Lorig et al. (2015b). We present the result here.

Theorem 5.4. *Let ψ_0 and ψ_n ($n \geq 1$) satisfy* (81) *and* (82)*, respectively. Then, omitting y-dependence for simplicity, we have*

$$\psi_0(t) = \exp\left((T-t)\frac{1-\gamma}{\eta\gamma}(\tfrac{1}{2}\lambda^2)_0 \right), \qquad \psi_n(t) = \widehat{\mathcal{L}}_n(t,T)\psi_0(t),$$

where the linear operator $\widehat{\mathcal{L}}_n(t,T)$ is given by

$$\widehat{\mathcal{L}}_n(t,T) := \sum_{k=1}^{n} \int_t^T \mathrm{d}t_1 \int_{t_1}^T \mathrm{d}t_2 \cdots \int_{t_{k-1}}^T \mathrm{d}t_k \sum_{I_{n,k}} \widehat{\mathcal{G}}_{i_1}(t,t_1)\widehat{\mathcal{G}}_{i_2}(t,t_2)\cdots\widehat{\mathcal{G}}_{i_k}(t,t_k),$$

with $I_{n,k}$ defined in (57) *and*

$$\widehat{\mathcal{G}}_i(t,t_k) = \widehat{\mathcal{A}}_i(\widehat{\mathcal{Y}}(t,t_k)), \qquad \widehat{\mathcal{Y}}(t,t_k) = y + (t_k - t)\left(c_0 + \frac{1-\gamma}{\gamma}(\rho\beta\lambda)_0 \right) + 2(\tfrac{1}{2}\beta^2)_0\partial_y.$$

Here, the notation $\widehat{\mathcal{A}}_i(\widehat{\mathcal{Y}}(t,t_k))$ indicates that y is replaced by $\widehat{\mathcal{Y}}(t,t_k)$ in the coefficients of $\widehat{\mathcal{A}}_i$.

Proof. See (Lorig et al., 2015b, Theorem 7). □

Having obtained an explicit expression for ψ_n ($n \geq 0$) we now define $\bar{\psi}_n$, the nth-order approximation of ψ:

$$\bar{\psi}_n := \sum_{k=0}^{n} \psi_k, \quad \text{with} \quad \bar{y} = y. \tag{83}$$

The accuracy of the series approximation $\bar{\psi}_n$ is established in Lorig et al. (2015a).

Theorem 5.5. *Let ψ be the solution of* (80) *with $a = 1$ and let $\bar{\psi}_n$ be defined by* (83) *with ψ_i ($i \geq 0$) as given in Theorem 5.4. Then, under Assumption 5.1, we have*

$$\sup_y |\psi(t,y) - \bar{\psi}_n(t,y)| = \mathcal{O}(\tau^{\frac{n+3}{2}}), \qquad \tau := T - t. \tag{84}$$

Proof. See (Lorig et al., 2015a, Theorem 3.10). □

Our task is now to translate the approximation $\bar{\psi}_n$ and accuracy result for $|\psi - \bar{\psi}_n|$ into an approximation $\bar{V}^{(n)}$ and accuracy result for $|V - \bar{V}^{(n)}|$. Expanding V^a, given by (76), in powers of a, we obtain

$$\begin{aligned} V^a &= \frac{w^{1-\gamma}}{1-\gamma}(\psi^a)^\eta \\ &= \frac{w^{1-\gamma}}{1-\gamma}\psi_0^\eta + \frac{w^{1-\gamma}}{1-\gamma}\sum_{k=1}^{\infty} a^k \left(\sum_{m=1}^{k} \frac{1}{m!}\left(\partial_\psi^m \psi_0^\eta\right) \left(\sum_{i \in I_{k,m}} \prod_{j=1}^{m} \psi_{i_j} \right) \right) =: \sum_{k=0}^{\infty} a^k V_k, \end{aligned}$$

where $I_{k,m}$ is defined in (57), and

$$\bar{V}^{(n)} = \sum_{k=0}^{n} V^{(k)}, \qquad V^{(0)} = \frac{w^{1-\gamma}}{1-\gamma}\psi_0^\eta, \qquad V^{(k)} = \frac{w^{1-\gamma}}{1-\gamma}\sum_{m=1}^{k}\frac{1}{m!}\left(\partial_\psi^m \psi_0^\eta\right)\left(\sum_{i \in I_{k,m}} \prod_{j=1}^{m} \psi_{i_j} \right), \tag{85}$$

and $\bar{y}$ is set by $\bar{y} = y$. The following theorem establishes the accuracy of $\bar{V}^{(n)}$, the nth-order approximation of the value function V.

Theorem 5.6. *Let (X, Y) have stochastic volatility dynamics* (75) *and assume the utility function U is of the power utility class* (67)*. Then, under Assumption 5.1, for a fixed w, the approximate value function $\bar{V}^{(n)}$, given by* (85)*, satisfies*

$$\sup_y |V(t,y,w) - \bar{V}^{(n)}(t,y,w)| = \mathcal{O}(\tau^{\frac{n+3}{2}}), \qquad \tau := T - t,$$

where η is defined in (76)*.*

Proof. Theorem 5.4 implies that $\psi_0(t) = \mathcal{O}(1)$ as $\tau \to 0$ and equation (84) implies

$$\sup_y \psi_n(t,y) = \mathcal{O}(\tau^{\frac{n+2}{2}}), \qquad n \geq 1.$$

It therefore follows from (85) that V_n satisfies

$$\sup_y V_n(t,y,w) = \mathcal{O}(\tau^{\frac{n+2}{2}}), \qquad n \geq 1.$$

Therefore, we have

$$\sup_y |V(t,y,w) - \bar{V}^{(n)}(t,y,w)| = \mathcal{O}(\tau^{\frac{n+3}{2}}),$$

as claimed. □

6 Examples

In this section we provide two numerical examples, which illustrate the accuracy and versatility of the series approximations developed in this paper. Both are based on power utility, but the first order approximations described in Section 3.7 could be computed for utility functions outside of this class, for instance mixture of power utilities, introduced in Fouque et al. (2013), which allow for wealth-varying relative risk aversion. There the solution of the constant parameter Merton problem M is computed numerically, and LSV corrections in the formulas of Section 3.7 can be obtained by numerical differentiation.

6.1 Stochastic volatility example

In our first example, we consider a stochastic volatility model in which the coefficients (μ, σ, c, β) appearing in (2) are given by

$$\mu(y) = \mu, \qquad \sigma(y) = \frac{1}{\sqrt{y}}, \qquad c(y) = \kappa(\theta - y), \qquad \beta(y) = \delta\sqrt{y}. \tag{86}$$

Here, the constants (κ, θ, δ) must satisfy the usual Feller condition: $2\kappa\theta \geq \delta^2$.

Assuming power utility (67), an explicit formula for the value function of the infinite horizon consumption problem is obtained in Chacko and Viceira (2005). For the terminal utility optimization problem that we consider in this paper, an explicit formula for the value function V in (3) is obtained in (Fouque et al., 2013, Section 6.4):

$$V(t,y,w) = \left(\frac{w^{1-\gamma}}{1-\gamma}\right) \mathrm{e}^{\eta A(T-t)y + \eta B(T-t)}, \qquad \eta = \frac{\gamma}{\gamma + (1-\gamma)\rho^2}, \tag{87}$$

$$A(t) = a_+ \frac{1 - \mathrm{e}^{-\alpha t}}{1 - (a_-/a_+)\mathrm{e}^{-\alpha t}}, \qquad B(t) = \kappa\theta \left(a_- t - \frac{2}{\delta^2} \log \left(\frac{1 - (a_-/a_+)\mathrm{e}^{-\alpha t}}{1 - (a_-/a_+)} \right) \right),$$

where

$$a_\pm = \frac{-q \pm \sqrt{q^2 - 4pr}}{2p}, \qquad \alpha = \sqrt{q^2 - 4pr},$$

$$p = \frac{1}{2}\delta^2, \qquad q = \delta \left(\frac{1-\gamma}{\gamma} \right) \mu\rho - \kappa, \qquad r = \frac{1}{2} \left(\frac{1-\gamma}{\eta\gamma} \right) \mu^2.$$

An explicit formula for the optimal investment strategy π^* can be obtained by inserting (87) into (6). Likewise, an explicit formula for the implied Sharpe ratio Λ can be obtained by inserting (87) into (74).

The zeroth, first and second order approximations for V, π^* and Λ can be obtained using the results of Section 5. Fixing $\bar{y} = y$ and using (68), (71), (72) and (73), we obtain

$$V^{(0)} = U(w) \exp \left(\frac{1-\gamma}{\gamma} \frac{\mu^2 y}{2} (T-t) \right), \quad V^{(1)} = \frac{1-\gamma}{4\gamma^2} \mu^2 (T-t)^2 \Big(\gamma\kappa(\theta - y) + (1-\gamma)\rho\delta\mu y \Big) V^{(0)},$$

$$\pi^* \approx \frac{w}{\gamma}\mu y + \frac{w}{\gamma}\rho\delta\mu^2 \frac{1-\gamma}{\gamma}(T-t)y,$$

$$\Lambda_0 = \mu\sqrt{y}, \qquad \Lambda_1 = \frac{(T-t)}{4\gamma\sqrt{y}} \mu \Big(\gamma\kappa(\theta - y) + (1-\gamma)\rho\delta\mu y \Big),$$

where in the strategy we have also expanded the coefficients in Taylor series. The second-order terms are omitted for the sake of brevity.

Let us examine the approximation for π^*, the optimal investment strategy. The first term $\mu y w/\gamma$ is the amount of money an investor would place in the risky asset if the volatility of X were frozen at $\sigma(y) = 1/\sqrt{y}$. The next term contains the adjustment for stochastic volatility effects. As volatility is empirically observed to be negatively correlated with price (the leverage effect), it is reasonable to assume that $\rho > 0$ (as $\sigma(y) = 1/\sqrt{y}$ is decreasing in y). Noting that $w\delta\mu^2 y/\gamma^2 > 0$, we see that the second term in π^* has the sign of $(1-\gamma)$. Thus, the more risk-averse an investor is (i.e., higher γ), the less of his wealth he will place in the risky asset X as a result of stochastic volatility effects. Indeed, for $\gamma > 1$, this term is negative, and in this case the investor reduces his holdings in the stock relative to the constant Merton case.

It is interesting to analyze how the approximation for the Merton implied Sharpe ratio depends on model parameters. The zeroth order approximation $\Lambda_0 = \mu\sqrt{y}$ is exactly the implied Sharpe ratio that one would obtain if volatility were frozen at $\sigma(y) = 1/\sqrt{y}$. Now, let us look at the expression for Λ_1. It is reasonable to assume (and therefore we do) that $\mu > 0$. If $Y_t = y < \theta$, then the instantaneous drift $\kappa(\theta - y)$ of Y will be positive. As Y moves upward the instantaneous volatility of X (given by $1/\sqrt{Y_t}$) decreases and the instantaneous Sharpe ratio $\mu\sqrt{Y_t}$ increases. As a result, we would expect the Merton implied Sharpe ratio to increase and this is captured by the first term in Λ_1. Similarly, if $Y_t = y > \theta$, the instantaneous Sharpe ratio would tend to decrease and we would expect the implied Sharpe ratio to decrease as well. Assuming $\rho > 0$ we see that the second term in Λ_1 has the sign of $(1-\gamma)$. Thus, the more risk-averse an investor is, the lower his Merton implied Sharpe ratio will be, and the reduction is magnified by δ, the volatility of the volatility factor. These observations about the optimal strategy and implied Sharpe ratio would not be obvious from the dynamics of (X, Y).

In Figure 1 we plot as a function of $\sigma = 1/\sqrt{y}$ the exact value function V, the exact optimal investment strategy π^* and the exact implied Sharpe ratio Λ. We also plot the zeroth, first and second-order approximations of these quantities. For all three quantities, we observe a close match between the exact function (u, π^* and Λ) and the second order approximation. Figure 1 also includes a plot of the implied Sharpe ratio (both exact Λ and the second order approximation) as a function of the risk-aversion parameter γ for three different time horizons. It is clear from the figure that the approximation is most accurate at the shortest time horizons, consistent with the accuracy result of Theorem 5.6.

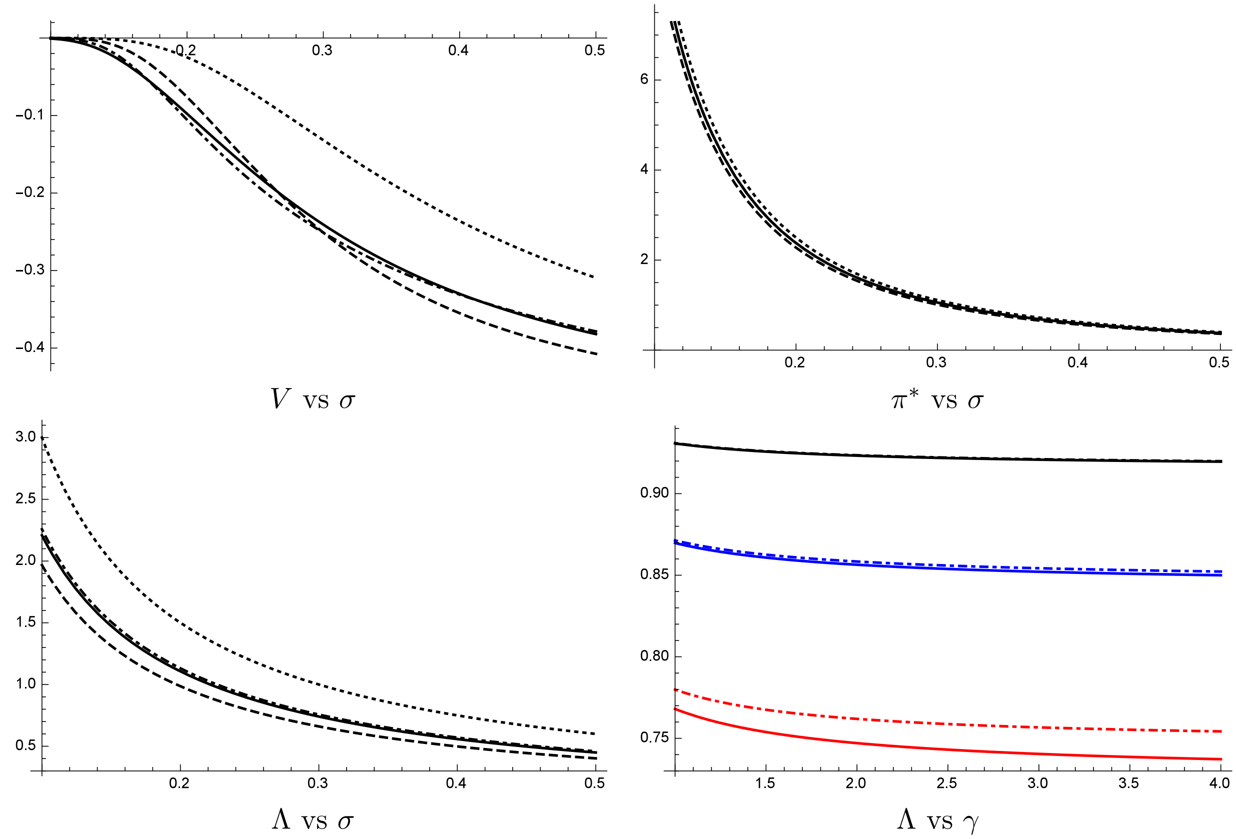

Figure 1: *The value function (top left), optimal strategy (top right) and implied Sharpe ratio (bottom left) are plotted as a function of instantaneous level of volatility* $\sigma = 1/\sqrt{y}$ *assuming power utility* (67) *and dynamics given by stochastic volatility model* (86). *In all three plots, the solid line corresponds to the exact function and the dotted, dashed and dot-dashed lines correspond to the zeroth, first and second-order approximations, respectively. The parameters used in these three plots are:* $T - t = 4.0$, $w = 1.0$, $\kappa = 0.3$, $\theta = 0.2$, $\delta = 0.3$, $\rho = 0.75$, $\mu = 0.3$ *and* $\gamma = 3.0$. *On the bottom right we fix the volatility* $\sigma = 0.3$ *and we plot the implied Sharpe ratio* Λ *as a function of* γ *for three different time horizons* $T - t = \{1, 2, 4\}$ *corresponding to black, blue and red, respectively. The solid lines are exact. The dot-dashed lines correspond to the second-order approximation. The parameters used in the bottom right plot are* $w = 1.0$, $\kappa = 0.3$, $\theta = 0.2$, $\delta = 0.3$, $\rho = -0.75$ *and* $\mu = 0.3$. *It is interesting to note that, with* $\rho > 0$, *which corresponds to negative correlation between* X *and* $\sigma(Y)$, *increasing the risk-aversion parameter* γ *leads to an decrease in the implied Sharpe ratio.*

6.2 Local volatility example (CEV)

We now consider a local volatility model in which the coefficients (μ, σ) appearing in (2) are given by

$$\mu(x) = \mu, \qquad \sigma(x) = \delta e^{\eta x}. \tag{88}$$

This is the constant elasticity of variance (CEV) model, written in the log-stock variable. Since Y plays no role in the dynamics of X, the coefficients c and β do not appear.

Assuming power utility (67), an explicit formula for the value function V in this setting is obtained in Darius (2005)

$$V(t,x,w) = \frac{w^{1-\gamma}}{1-\gamma}\left(f(t,\mathrm{e}^{-2\eta x})\right)^{\gamma}, \qquad f(t,d) = A(t)\mathrm{e}^{B(t)d}, \quad (89)$$

$$A(t) = \mathrm{e}^{\lambda_+\eta(2\eta+1)(T-t)}\left(\frac{\lambda_- - \lambda_+}{\lambda_- - \lambda_+\mathrm{e}^{2\eta^2(\lambda_+-\lambda_-)(T-t)}}\right)^{\frac{2\eta+1}{2\eta}}, \qquad B(t) = \frac{1}{\delta^2}I(t),$$

$$I(t) = \frac{\lambda_+\left(1-\mathrm{e}^{2\eta^2(\lambda_+-\lambda_-)(T-t)}\right)}{1-(\lambda_+/\lambda_-)\mathrm{e}^{2\eta^2(\lambda_+-\lambda_-)(T-t)}}, \qquad \lambda_\pm = \frac{\mu \pm \sqrt{\gamma\mu^2}}{2\eta\gamma}.$$

The optimal investment strategy π^* can be obtained by inserting (89) into (6) An explicit expression for the implied Sharpe ratio Λ can be obtained by inserting (89) into (74).

The zeroth, first and second order approximations for V, π^* and Λ can be obtained using the results of Section 5. Fixing $\bar{x} = x$ and using (68), (71), (72) and (73), we obtain

$$V^{(0)} = U(w)\exp\left(\frac{1-\gamma}{2\gamma}\frac{\mu^2}{\delta^2}\mathrm{e}^{-2\eta x}(T-t)\right), \quad V^{(1)} = \frac{-(1-\gamma)(T-t)^2\eta\mu^2}{2\gamma^2\delta^2}\mathrm{e}^{-2\eta x}\left(\mu - \gamma\tfrac{1}{2}\delta^2\mathrm{e}^{2\eta x}\right)u_0,$$

$$\pi^* \approx \left(1 - \frac{\mu\eta(1-\gamma)(T-t)}{\gamma}\right)\frac{\mu}{\gamma(\delta\mathrm{e}^{\eta x})^2}w,$$

$$\Lambda_0 = \frac{\mu}{\delta\mathrm{e}^{\eta x}}, \qquad \Lambda_1 = \frac{-(T-t)}{2}\frac{\eta\mu}{\delta\mathrm{e}^{\eta x}}\left(\frac{\mu}{\gamma} - \tfrac{1}{2}\delta^2\mathrm{e}^{2\eta x}\right).$$

The second-order terms are omitted for the sake of brevity.

We first comment on the expression for π^*. As usual, the first term in the expression for π^* corresponds to the amount of money an investor would place in the risky asset X if volatility were frozen at $\sigma(x) = \delta\mathrm{e}^{\eta x}$. In order to be consistent with the leverage effect, it is reasonable to assume $\eta < 0$. We also assume $\mu > 0$. This being the case, all investors will increase their holdings in X as the value of the risky asset moves upwards as a result of the corresponding decrease in instantaneous volatility. The second term in the expression for π^* has the sign of $-\eta(1-\gamma)$. Thus, the more risk-averse an investor (i.e., higher γ) the more he will reduce his holdings in the risky asset as a result of local volatility effects. Indeed, for $\gamma > 1$, which consistend with oberved investor behavior, this term is negative, indicating a reduction in the stock holding compared to the frozen Merton proportion.

Now, let us examine the expressions for Λ_0 and Λ_1, continuing with the assumption that $\eta < 0$. As X_t moves upward the instantaneous volatility $\sigma(X_t) = \delta\exp(\eta X_t)$ moves downward and the instantaneous Sharpe ratio $\mu\delta\exp(-\eta X_t)$ moves upward. As a result, as X_t increases we would expect a higher implied Sharpe ratio, and this is captured by Λ_0. The correction term Λ_1 has the sign of $(\mu/\gamma - \delta^2\mathrm{e}^{2\eta x}/2)$. Thus, the first order correction to the implied Sharpe ratio is an interplay between the drift μ, which raises Λ_1 and the local variance $\delta^2\mathrm{e}^{2\eta x}$, which lowers Λ_1. The relative weight of these two terms is modulated by the risk-aversion parameter γ, with a higher γ placing a less emphasis on the drift.

In Figure 2, we plot as a function of $\sigma = \delta\mathrm{e}^{\eta x}$ the value function V, the optimal investment strategy π^* and the implied Sharpe ratio Λ. We also plot the zeroth, first and second order approximations of these quantities. For all three quantities, we observe a close match between the exact functions (V, π^* and Λ) and their second order approximation. Figure 2 also contain plots of the implied Sharpe ratio Λ (both exact Λ and the second order approximation $\bar{\Lambda}_2$) as a function of the risk-aversion parameter γ for three different time horizons.

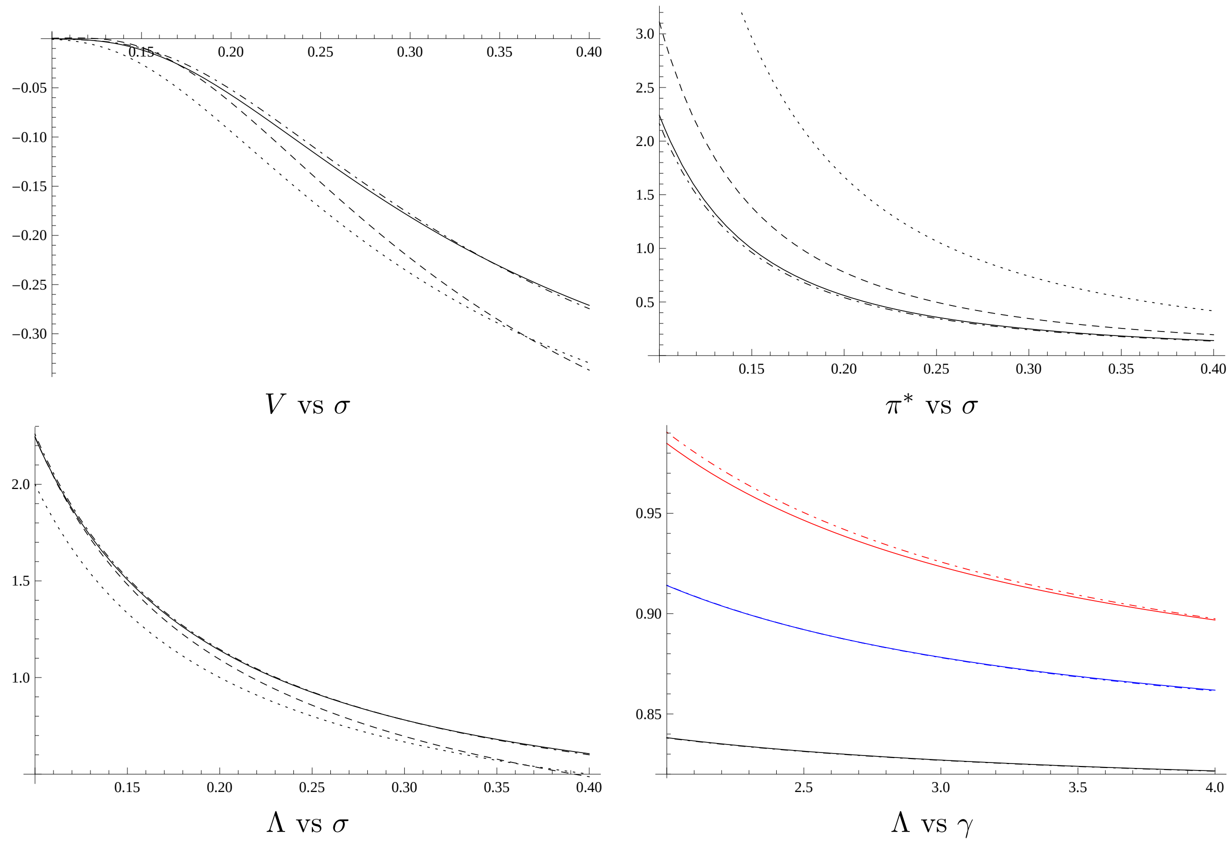

Figure 2: *Value function (top left), optimal strategy (top right) and implied Sharpe ratio (bottom left) are plotted as a function of* $\sigma = \delta e^{\eta x}$ *assuming power utility* (67) *and dynamics given by local volatility model* (88). *In all three plots, the solid line corresponds to the exact function, and the dotted, dashed and dot-dashed lines correspond to the zeroth, first and second-order approximations, respectively. The parameters used in these three plots are:* $T - t = 5$, $w = 1.0$, $\eta = -0.8$, $\delta = 0.3$, $\mu = 0.3$ *and* $\gamma = 3.00$. *On the bottom right we fix the volatility* $\sigma = 0.25$ *and we plot the implied Sharpe ratio* Λ *as a function of* γ *for three different time horizons* $T - t = \{1, 3, 5\}$ *corresponding to black, blue and red, respectively. The solid lines are exact. The dot-dashed lines are our second order approximation. The parameters used in the bottom right plot are:* $w = 1.0$, $\eta = -0.8$, $\delta = 0.3$ *and* $\mu = 0.3$. *Note that with* $\eta < 0$ *(which is consistent with the leverage effect) a larger risk-aversion parameter* γ *results in a lower implied Sharpe ratio.*

7 Conclusion

In this paper we consider the finite horizon utility maximization problem in a general LSV setting. Using polynomial expansion methods, we obtain an approximate solution for the value function and optimal investment strategy. The zeroth-order approximation of the value function and optimal investment strategy correspond to those obtained by Merton (1969) when the risky asset follows a geometric Brownian motion.

The first-order term in the value function approximation can always be expressed as a differential operator acting on the zeroth-order term. Higher-order corrections can always be expressed as a nonlinear transformation of a convolution with a Gaussian kernel. For certain utility functions, these convolutions can be expressed in closed-form as a differential operator acting on the zeroth-order term. Corrections to the zeroth-order optimal investment strategy can be obtained from the approximation of the value function.

We also introduce in this paper the concept of an implied Sharpe ratio and derive an approximation for this quantity. We obtain specific results for power utility and give a rigorous error bound for the value function in a stochastic volatility setting. Finally, we provide two numerical examples to illustrate the accuracy and versatility of our approach. These examples enable us to give financial interpretations of the approximation formulas. The expansion techniques presented in this paper naturally lend themselves to other nonlinear stochastic control problems. Recent results for indifference pricing of options contracts have been developed in Lorig (2016).

References

Chacko, G. and L. M. Viceira (2005). Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets. *Review of Financial Studies 18*(4), 1369–1402.

Corielli, F., P. Foschi, and A. Pascucci (2010). Parametrix approximation of diffusion transition densities. *SIAM Journal on Financial Mathematics 1*(1), 833–867.

Darius, D. (2005). The constant elasticity of variance model in the framework of optimal investment problems. *Thesis: Princeton University*.

Fleming, W. H. and H. M. Soner (1993). *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag.

Fouque, J.-P., G. Papanicolaou, and R. Sircar (2000). *Derivatives in Financial Markets with Stochastic Volatility*. Cambridge University Press.

Fouque, J.-P., G. Papanicolaou, R. Sircar, and K. Solna (2011). *Multiscale stochastic volatility for equity, interest rate, and credit derivatives*. Cambridge University Press.

Fouque, J.-P., R. Sircar, and T. Zariphopoulou (2013). Portfolio optimization and stochastic volatility asymptotics. *Mathematical Finance*. To appear.

Jonsson, M. and R. Sircar (2002a). Optimal investment problems and volatility homogenization approximations. In A. Bourlioux, M. Gander, and G. Sabidussi (Eds.), *Modern Methods in Scientific Computing and Applications*, Volume 75 of *NATO Science Series II*, pp. 255–281. Kluwer.

Jonsson, M. and R. Sircar (2002b, October). Partial hedging in a stochastic volatility environment. *Mathematical Finance 12*(4), 375–409.

Lorig, M. (2016). Indifference prices and implied volatilities. *Mathematical Finance, (to appear)*.

Lorig, M., S. Pagliarani, and A. Pascucci (2014). Asymptotics for d-dimensional Lévy-type processes. In *Large Deviations and Asymptotic Methods in Finance*, pp. 321–344. Springer.

Lorig, M., S. Pagliarani, and A. Pascucci (2015a). Analytical expansions for parabolic equations. *SIAM Journal on Applied Mathematics 75*, 468–491.

Lorig, M., S. Pagliarani, and A. Pascucci (2015b). Explicit implied volatilities for multifactor local-stochastic volatility models. *Mathematical Finance*.

Lorig, M., S. Pagliarani, and A. Pascucci (2015c). A family of density expansions for Lévy-type processes with default. *Annals of Applied Probability 25*(1), 235–267.

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *Review of Economics and statistics 51*(3), 247–257.

Pagliarani, S. and A. Pascucci (2012). Analytical approximation of the transition density in a local volatility model. *Cent. Eur. J. Math. 10*(1), 250–270.

Pagliarani, S., A. Pascucci, and C. Riga (2013). Adjoint expansions in local Lévy models. *SIAM J. Financial Math. 4*, 265–296.

Pham, H. (2009). *Continuous-time Stochastic Control and Optimization with Financial Applications*. Springer.

Zariphopoulou, T. (2001). A solution approach to valuation with unhedgeable risks. *Finance and Stochastics 5*(1), 61–82.