Portfolio Optimization & Stochastic Volatility Asymptotics

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Abstract

We study the Merton portfolio optimization problem in the presence of stochastic volatility using asymptotic approximations when the volatility process is characterized by its time scales of fluctuation. This approach is tractable because it treats the incomplete markets problem as a perturbation around the complete market constant volatility problem for the value function, which is well-understood. When volatility is fast mean-reverting, this is a singular perturbation problem for a nonlinear Hamilton-Jacobi-Bellman PDE, while when volatility is slowly varying, it is a regular perturbation. These analyses can be combined for multifactor multiscale stochastic volatility models. The asymptotics shares remarkable similarities with the linear option pricing problem, which follows from some new properties of the Merton risk-tolerance function.

We give examples in the family of mixture of power utilities and also we use our asymptotic analysis to suggest a "practical" strategy which does not require tracking the fast-moving volatility. In this paper, we present formal derivations of asymptotic approximations, and we provide a convergence proof in the case of power utility and single factor stochastic volatility. We assess our approximation in a particular case where there is an explicit solution.

1   Introduction

The Merton problem of portfolio optimization in continuous-time stochastic models has a long history dating to the seminal papers by Robert Merton published in 1969 and 1971 and re-printed in Merton [1992]. There, he was able to produce explicit solutions for how to allocate investment capital between risky stocks and a riskless money-market account, when the stocks are modeled as geometric Brownian motions (that is, they have constant volatilities), and when the utility function that describes the investor's risk-aversion is of some specific types.

The goal of this article is to study the optimal investment problem within multiscale stochastic volatility models. In this context, asymptotic analysis has been developed over a number of years to simplify *option pricing* problems, and this is described in the recent book by Fouque et al. [2011], where singular and regular perturbation methods can be used for effective approximations of the linear pricing problem; here we present new results for the nonlinear Merton problem for *general utility functions* on $\boldsymbol{R}_+$. This extends earlier analysis for simple power utilities in Fouque et al. [2000, Section 10.1], and expansions for the partial hedging problem in Jonsson and Sircar [2002a,b] in the dual problem, both for *fast mean-reverting* stochastic volatility. Here, we construct the expansion directly in the primal problem under both fast and

slow volatility fluctuations. Indifference pricing approximations with *exponential utility* and fast volatility were studied in Sircar and Zariphopoulou [2005].

We work under the multiscale stochastic volatility framework used in Fouque et al. [2011] for option pricing, where there is one fast volatility factor, and one slow. Here, the volatility is a function σ of a fast factor Y and a slow factor Z: $\sigma(Y_t, Z_t)$. We also allow these two factors to drive the growth rate: $\mu(Y_t, Z_t)$. The stock or index price process S and its volatility-driving factors (Y, Z) are described by:

$$dS_t = \mu(Y_t, Z_t)S_t\,dt + \sigma(Y_t, Z_t)S_t\,dW_t^{(0)}, \tag{1}$$

$$dY_t = \frac{1}{\varepsilon}\,b(Y_t)\,dt + \frac{1}{\sqrt{\varepsilon}}\,a(Y_t)\,dW_t^{(1)},$$

$$dZ_t = \delta\,c(Z_t)\,dt + \sqrt{\delta}\,g(Z_t)\,dW_t^{(2)},$$

where the standard Brownian motions $\left(W_t^{(0)}, W_t^{(1)}, W_t^{(2)}\right)$ are correlated as follows:

$$d\langle W^{(0)}, W^{(1)}\rangle_t = \rho_1\,dt, \qquad d\langle W^{(0)}, W^{(2)}\rangle_t = \rho_2\,dt, \qquad d\langle W^{(1)}, W^{(2)}\rangle_t = \rho_{12}\,dt,$$

where $|\rho_1| < 1$, $|\rho_2| < 1$, $|\rho_{12}| < 1$, and $1 + 2\rho_1\rho_2\rho_{12} - \rho_1^2 - \rho_2^2 - \rho_{12}^2 > 0$, in order to ensure positive definiteness of the covariance matrix of the three Brownian motions. The model is described by the coefficients μ, σ, b, a, c and g. The parameters ε and δ, when small, characterize the fast and slow variation of Y and Z factors respectively. Further technical details of the model are presented during the formal asymptotic calculations in the following sections.

As alluded to in Chacko and Viceira [2005] for instance, two volatility factors, one fast and one slow, need to be considered simultaneously, but the existing literature handles models with only one volatility component. Typically these results are applied to the effects of the *slow* factor. In Chacko and Viceira [2005], at the end of their Section 5.1, the authors explicitly state:

"The estimate of the reversion parameter κ in the precision equation implies a half-life of a shock to precision of about 2 years in the monthly sample. The rate of mean reversion is slower in the annual sample, where the estimate of the half-life of a shock to precision is about 16 years. French, Schwert and Stambaugh (1987), Schwert (1989), and Campbell and Hentschel (1990) have also found a relatively slow speed of adjustment of shocks to stock volatility in low frequency data. This slow reversion to the mean in low frequency data contrasts with the fast speed of adjustment detected in high frequency data by Andersen, Benzoni and Lund (1998).

These results suggest that there might be high frequency and low frequency (or long-memory) components in stock market volatility (Chacko and Viceira, 2003). By construction, the single component model ... cannot capture these components simultaneously. On the other hand, it is very difficult to find analytical solutions for a model with multiple components in volatility. We hope that by focusing on estimates of the single component model derived from low frequency data, we can capture the persistence and variability characteristics of the volatility process that are most relevant to long-term investors. Accordingly, in our calibration exercise we focus on the monthly and annual estimates of the single component model."

The main contribution of our work is to be able to treat the portfolio optimization problem with general utility functions allowing for non-constant relative risk aversion (in contrast to the case of power utilities), and in the context of incomplete markets with stochastic volatility. We achieve this by driving volatility with two factors, one on a fast time scale and one on a slow time scale, and using perturbation methods. Surprisingly, the first corrections in the expansion of the value function are given explicitly in terms of the derivatives of the leading order value function (itself the solution of the Hamilton-Jacobi-Bellman (HJB) PDE for the Merton problem with constant parameters). The first order correction to the optimal strategy is also given explicitly in terms of derivatives of the leading order value function. Of course this is an enormous gain as the constant coefficient problem involves a PDE in time plus one space dimension, whereas the original problem has an HJB equation with time plus three space dimensions.

We also show that in the leading order term in the portfolio allocation, the fast factor is averaged in some parts of the formula, but tracked in others, and the corresponding strategy achieves the optimal expected utility up to the first order corrections. The asymptotic analysis here for the nonlinear portfolio problem has remarkable similarities with that for the linear European option pricing problem, thanks to the properties of a specific "risk-tolerance" function, specifically that it satisfies Black's (fast diffusion) PDE.

To keep the presentation manageable, we focus on the analysis of the two factors *separately*. We begin in Section 2 with the case of fast mean-reverting stochastic volatility, which leads to a singular perturbation problem for the associated HJB PDE. In Section 3, we analyze the case of slowly fluctuating volatility, which leads to a regular perturbation problem, and reveals a useful "Vega-Gamma" relationship for the classical Merton value function. Section 4 discusses how the fast and slow results can be combined for approximations under multiscale stochastic volatility.

Section 5 proposes a "practical" portfolio strategy for the multifactor mutiscale volatility model. The advantage of this strategy is that it does not require tracking the fast volatility factor, and we quantify its suboptimality. In Section 6.2, we introduce the family of mixture of power utility functions, which allows for non-constant relative risk aversion, specifically, declining with increasing wealth. We present numerical solutions to illustrate the tractability of the asymptotic approximations. We give an accuracy proof for power utilities and one-factor stochastic volatility (either fast or slow) in Section 6.3. In Section 6.4, we compare our approximation within a model with explicit solution. Section 7 concludes and suggests directions of extension.

2   Merton Problem under Fast Mean-Reverting Stochastic Volatility

We first analyze the Merton problem over a finite time horizon $[0, T]$ with general terminal utility function under fast mean-reverting stochastic volatility. We have the following dynamics for a stock or index price process S:

$$dS_t = \mu(Y_t)S_t \, dt + \sigma(Y_t)S_t \, dW_t^{(0)} \tag{2}$$

$$dY_t = \frac{1}{\varepsilon}b(Y_t) \, dt + \frac{1}{\sqrt{\varepsilon}}a(Y_t) \, dW_t^{(1)}, \tag{3}$$

where $W^{(0)}$ and $W^{(1)}$ are Brownian motions in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_t, \boldsymbol{P})$ with instantaneous correlation coefficient between volatility and stock price shocks $\rho_1 \in (-1, 1)$. Here we assume that the process $Y_t = Y_{t/\varepsilon}^{(1)}$ in distribution, where $Y^{(1)}$ is an ergodic diffusion process with unique invariant distribution Φ, independent of ε. This re-scaling of time explains the $\frac{1}{\sqrt{\varepsilon}}$ in front of the Brownian motion in (3) and means that Y also has unique invariant distribution Φ.

We use the notation $\langle \cdot \rangle$ for averaging with respect to Φ:

$$\langle g \rangle = \int g(y)\Phi(dy). \tag{4}$$

The ergodicity models mean-reversion of volatility, while the parameter $\varepsilon > 0$ characterizes the typical time-scale over which it returns to its long-run mean level. We are interested in the fast mean-reverting regime, that is as $\varepsilon \downarrow 0$. For a detailed exposition of stochastic volatility time scales, we refer to [Fouque et al., 2011, Chapter 3]. For a general reference on the extensive literature on the Merton problem under various market frictions, particularly by dynamic programming methods, we refer to the recent book by Pham [2009].

Let $(X_t, t \in [0, T])$ denote the wealth process and π_t the amount of wealth an investor holds in stock at time t, with the remaining held in a money market account paying interest at rate r. With continuous self-financing trading, we have $dX_t = \pi_t \frac{dS_t}{S_t} + r(X - \pi_t) \, dt$, that is

$$dX_t = (rX_t + \pi_t(\mu(Y_t) - r)) \, dt + \pi_t\sigma(Y_t) \, dW_t^{(0)}.$$

For simplicity of exposition and without loss of generality, we will take $r = 0$ throughout.

Assumption 2.1. The investor has a terminal utility function $U(x)$ on R_+, which is smooth: $U \in C^\infty(R_+)$. It is also strictly increasing and strictly concave, and it satisfies the "usual conditions" (Inada and Asymptotic Elasticity):

$$U'(0^+) = \infty, \qquad U'(\infty) = 0, \qquad AE[U] := \lim_{x \to \infty} x \frac{U'(x)}{U(x)} < 1.$$

Assumption 2.1 holds throughout the paper and will not always be recalled.

We define the value function

$$V^\varepsilon(t, x, y) = \sup_\pi E \{U(X_T) \mid X_t = x, Y_t = y\},$$

where the supremum is taken over admissible strategies that are $\mathcal{F}_t$-progressively measurable and satisfy

$$E \left\{ \int_0^T \sigma(Y_t)^2 \pi_t^2 dt \right\} < \infty.$$

The associated Hamilton-Jacobi-Bellman (HJB) PDE problem for V^ε is

$$V_t^\varepsilon + \frac{1}{\varepsilon} \mathcal{L}_0 V^\varepsilon + \max_\pi \left(\frac{1}{2} \sigma(y)^2 \pi^2 V_{xx}^\varepsilon + \pi \left(\mu(y) V_x^\varepsilon + \frac{\rho_1 a(y) \sigma(y)}{\sqrt{\varepsilon}} V_{xy}^\varepsilon \right) \right) = 0, \qquad t < T, x \in R_+, y \in R, \quad (5)$$

with the terminal condition $V^\varepsilon(T, x, y) = U(x)$, and where $\mathcal{L}_0$ is the infinitesimal generator of the process $Y^{(1)}$:

$$\mathcal{L}_0 = \frac{1}{2} a(y)^2 \frac{\partial^2}{\partial y^2} + b(y) \frac{\partial}{\partial y}. \tag{6}$$

Maximizing the quadratic expression in π, the optimal portfolio function is given in feedback form by

$$\pi^*(t, x, y) = -\frac{\mu(y)}{\sigma(y)^2} \frac{V_x^\varepsilon}{V_{xx}^\varepsilon} - \frac{\rho_1 a(y)}{\sqrt{\varepsilon} \sigma(y)} \frac{V_{xy}^\varepsilon}{V_{xx}^\varepsilon}. \tag{7}$$

Inserting the maximizer π^* into (5) leads to the HJB PDE problem

$$V_t^\varepsilon + \frac{1}{\varepsilon} \mathcal{L}_0 V^\varepsilon - \frac{\left(\lambda(y) V_x^\varepsilon + \frac{\rho_1 a(y)}{\sqrt{\varepsilon}} V_{xy}^\varepsilon \right)^2}{2 V_{xx}^\varepsilon} = 0, \qquad t < T, x \in R_+, y \in R, \tag{8}$$

$$V^\varepsilon(T, x, y) = U(x), \tag{9}$$

where μ appears through the Sharpe ratio λ, which is defined by $\lambda(y) = \frac{\mu(y)}{\sigma(y)}$.

Assumption 2.2. The value function $V^\varepsilon(t, x, y)$ is smooth on $[0, T] \times R_+ \times R$, and is strictly increasing and strictly concave in the wealth argument x for each $y \in R$ and $t \in [0, T)$. Moreover, it is the unique solution in this class of the HJB equation (8) with terminal condition (9).

We observe that (8) is a fully nonlinear PDE which is not easily solved either analytically or numerically, for general utility functions. In the limit $\varepsilon \to 0$, it is a singular perturbation problem, and our approach is to construct an asymptotic approximation of the solution. This approach has been used to simplify a host of derivative pricing problems which are characterized by linear PDEs as detailed in Fouque et al. [2011], and summarized later in Section 2.4. There, the challenges involve non-smoothness of option payoffs and boundary conditions for non-European options. Here, the terminal condition is smooth but the main challenge is the nonlinearity of the PDE.

2.1 Classical Constant Parameter Merton Problem

The special case of μ and σ in (2) being constants is the classical Merton problem, which has been studied extensively. In the upcoming asymptotic analysis of the stochastic parameter problem, the classical Merton value function will play a key role in constructing the approximation to V^ε. To facilitate the presentation, we review some background results. To this end, we denote by $M(t, x; \lambda)$ the Merton value function when the growth rate μ, the volatility σ and hence the Sharpe ratio $\lambda = \mu/\sigma$, are *constant*, and the investor has utility function U satisfying our standing Assumption 2.1.

Then M is the unique smooth (on $[0, T] \times R_+$), strictly increasing and strictly concave solution of the HJB PDE problem

$$M_t - \frac{1}{2}\lambda^2 \frac{M_x^2}{M_{xx}} = 0, \qquad M(T, x; \lambda) = U(x). \tag{10}$$

It is also continuously differentiable with respect to λ. These properties have been established primarily using the Fenchel-Legendre transformation, which transforms (10) to a linear constant coefficient parabolic PDE problem (the heat equation), for which regularity results are standard.

It is also convenient to introduce the so-called *risk-tolerance function* $R(t, x; \lambda)$ associated with the classical Merton value function:

$$R(t, x; \lambda) = -\frac{M_x(t, x; \lambda)}{M_{xx}(t, x; \lambda)}, \tag{11}$$

which is well-defined as M is strictly concave. It follows from the smoothness of M in (t, x) that R is also smooth in (t, x). The classical Merton portfolio policy is given by $\pi^{(M)} = \frac{\lambda}{\sigma}R(t, x; \lambda)$.

We also define the differential operators

$$D_k = R(t, x; \lambda)^k \frac{\partial^k}{\partial x^k}, \qquad k = 1, 2, \cdots, \tag{12}$$

and the linear operator $\mathcal{L}_{t,x}(\lambda)$

$$\mathcal{L}_{t,x}(\lambda) = \frac{\partial}{\partial t} + \frac{1}{2}\lambda^2 D_2 + \lambda^2 D_1, \tag{13}$$

whose coefficients depend on $R(t, x; \lambda)$. We observe that the Merton PDE (10) can be re-written as

$$\mathcal{L}_{t,x}(\lambda)M = 0, \tag{14}$$

because

$$M_t - \frac{1}{2}\lambda^2 \frac{(M_x)^2}{M_{xx}} = M_t + \frac{1}{2}\lambda^2 \left(\frac{M_x}{M_{xx}}\right)^2 M_{xx} + \lambda^2 \left(-\frac{M_x}{M_{xx}}\right)M_x = \mathcal{L}_{t,x}(\lambda)M.$$

Our reason for introducing these operator notations will become clear in the derivation of the asymptotic expansions.

We will develop some further results about the constant λ Merton value and risk-tolerance functions as we need them for the asymptotic analysis, specifically the fast diffusion equation (Lemma 2.3) and the "Vega-Gamma" relationship (Lemma 3.2).

2.2 Expansion of the Value Function

We look for an expansion of the value function of the form

$$V^\varepsilon(t, x, y) = v^{(0)}(t, x, y) + \sqrt{\varepsilon}\, v^{(1)}(t, x, y) + \varepsilon v^{(2)}(t, x, y) + \varepsilon^{3/2} v^{(3)}(t, x, y) \cdots.$$

Inserting this expansion into (8), and collecting terms in successive powers of ε, we obtain at the highest order ε^{-1}:

$$\mathcal{L}_0 v^{(0)} - \frac{1}{2}\rho_1^2 a(y)^2 \frac{(v_{xy}^{(0)})^2}{v_{xx}^{(0)}} = 0.$$

As $\mathcal{L}_0$ takes derivatives in y, this equation is satisfied by $v^{(0)}(t, x)$ independent of y. With this choice, we have $v_y^{(0)} = 0$, and so expanding the nonlinear term in (8) up to order $\sqrt{\varepsilon}$ gives:

$$\frac{\left(\lambda(y)V_x^{\varepsilon} + \frac{\rho_1 a(y)}{\sqrt{\varepsilon}}V_{xy}^{\varepsilon}\right)^2}{2V_{xx}^{\varepsilon}} = \left(\lambda(y)(v_x^{(0)} + \sqrt{\varepsilon}v_x^{(1)}) + \rho_1 a(y)(v_{xy}^{(1)} + \sqrt{\varepsilon}v_{xy}^{(2)})\right)^2 \frac{1}{2v_{xx}^{(0)}}\left(1 - \sqrt{\varepsilon}\frac{v_{xx}^{(1)}}{v_{xx}^{(0)}}\right) + \cdots. \quad (15)$$

Therefore, at the next highest order $\varepsilon^{-1/2}$ in the expansion of the PDE, there is no contribution from the nonlinear term, and we obtain simply $\mathcal{L}_0 v^{(1)} = 0$. Again, we satisfy this equation with $v^{(1)} = v^{(1)}(t, x)$, independent of y.

Then, collecting the order one terms leads to:

$$v_t^{(0)} + \mathcal{L}_0 v^{(2)} - \frac{1}{2}\lambda(y)^2 \frac{(v_x^{(0)})^2}{v_{xx}^{(0)}} = 0. \quad (16)$$

2.2.1 Zeroth Order Term $v^{(0)}$

Equation (16) is a Poisson equation for $v^{(2)}$ whose solvability condition (Fredholm Alternative) requires that

$$\left\langle v_t^{(0)} - \frac{1}{2}\lambda(y)^2 \frac{(v_x^{(0)})^2}{v_{xx}^{(0)}} \right\rangle = 0,$$

where $\langle \cdot \rangle$ was defined in (4). Introducing the constant square-averaged Sharpe ratio $\bar{\lambda}$ by

$$\bar{\lambda}^2 = \left\langle \frac{\mu^2}{\sigma^2} \right\rangle, \quad (17)$$

and, as $v^{(0)}$ does not depend on y, the solvability condition gives

$$v_t^{(0)} - \frac{1}{2}\bar{\lambda}^2 \frac{(v_x^{(0)})^2}{v_{xx}^{(0)}} = 0, \quad (18)$$

and the terminal condition is $v^{(0)}(T, x) = U(x)$.

We see that (18) is the nonlinear PDE (10) for the Merton problem with constant Sharpe ratio $\bar{\lambda}$, and so,

$$v^{(0)}(t, x) = M(t, x; \bar{\lambda}), \quad (19)$$

where M is the classical constant parameter Merton value function introduced in Section 2.1.

Using the notation introduced in (13), we have that

$$\mathcal{L}_{t,x}(\bar{\lambda}) = \frac{\partial}{\partial t} + \frac{1}{2}\bar{\lambda}^2 D_2 + \bar{\lambda}^2 D_1 \quad \text{and} \quad D_k = \left(-\frac{v_x^{(0)}(t, x)}{v_{xx}^{(0)}(t, x)}\right)^k \frac{\partial^k}{\partial x^k} = R(t, x; \bar{\lambda})^k \frac{\partial^k}{\partial x^k}. \quad (20)$$

Then equation (18) can be re-written as

$$\mathcal{L}_{t,x}(\bar{\lambda})v^{(0)} = 0, \quad (21)$$

as shown in the derivation of (14).

6

2.2.2 First Order Term $v^{(1)}$

Similarly, we can write (16) as

$$\mathcal{L}_0 v^{(2)} + \mathcal{L}_{t,x}(\lambda(y))v^{(0)} = 0. \tag{22}$$

From (22) and (21), we have

$$\mathcal{L}_0 v^{(2)} = -\left(\mathcal{L}_{t,x}(\lambda(y)) - \mathcal{L}_{t,x}(\overline{\lambda})\right)v^{(0)} = -\left(\lambda(y)^2 - \overline{\lambda}^2\right)\left(\frac{1}{2}D_2 + D_1\right)v^{(0)}.$$

This is a Poisson equation for $v^{(2)}$ whose solutions that are in $L_2(\Phi)$ (ensuring reasonable behavior at infinity) differ by a constant (see, for instance, [Fouque et al., 2011, Section 3.2]). Therefore,

$$v^{(2)} = -\theta(y)\left(\frac{1}{2}D_2 + D_1\right)v^{(0)} + C(t, x), \tag{23}$$

where $\theta(y)$ is a solution of the ODE (in the y variable)

$$\mathcal{L}_0\theta = \left(\lambda(y)^2 - \overline{\lambda}^2\right), \tag{24}$$

and $C(t, x)$ is some 'constant' of integration in y, that may depend on (t, x).

Consequently, the expansion of the nonlinear term up to order $\sqrt{\varepsilon}$ computed in (15) simplifies to:

$$\frac{\left(\lambda(y)V_x^\varepsilon + \frac{\rho_1 a(y)}{\sqrt{\varepsilon}}V_{xy}^\varepsilon\right)^2}{2V_{xx}^\varepsilon} = \left(\lambda(y)(v_x^{(0)} + \sqrt{\varepsilon}v_x^{(1)}) + \frac{1}{2}\sqrt{\varepsilon}\rho_1 a(y)\theta'(y)\frac{\partial}{\partial x}\left(\frac{(v_x^{(0)})^2}{v_{xx}^{(0)}}\right)\right)^2 \frac{1}{2v_{xx}^{(0)}}\left(1 - \sqrt{\varepsilon}\frac{v_{xx}^{(1)}}{v_{xx}^{(0)}}\right) + \cdots,$$

and we observe that $C(t, x)$ no longer appears because $v^{(2)}$ appeared in (15) as $v_{xy}^{(2)}$.

Therefore, at order $\sqrt{\varepsilon}$ in the expansion of the PDE, we have

$$\mathcal{L}_0 v^{(3)} + v_t^{(1)} - \frac{1}{2v_{xx}^{(0)}}\left(2\lambda(y)v_x^{(0)}\left[\lambda(y)v_x^{(1)} + \frac{1}{2}\rho_1 a(y)\theta'(y)\frac{\partial}{\partial x}\left(\frac{(v_x^{(0)})^2}{v_{xx}^{(0)}}\right)\right] - \lambda(y)^2(v_x^{(0)})^2\frac{v_{xx}^{(1)}}{v_{xx}^{(0)}}\right) = 0.$$

Using the operator notations D_k and $\mathcal{L}_{t,x}$ in (20), the equation above can be re-arranged as:

$$\mathcal{L}_0 v^{(3)} + \mathcal{L}_{t,x}(\lambda(y))v^{(1)} - \frac{1}{2}\rho_1\lambda(y)a(y)\theta'(y)D_1^2 v^{(0)} = 0. \tag{25}$$

Equation (25) is a Poisson equation for $v^{(3)}$ whose solvability condition is

$$\mathcal{L}_{t,x}(\overline{\lambda})v^{(1)} = \frac{1}{2}\rho_1 B D_1^2 v^{(0)}, \tag{26}$$

where we define the constant

$$B = \langle \lambda a\theta'\rangle. \tag{27}$$

As $v^{(0)}$ already satisfies the terminal condition for the full problem, that is $v^{(0)}(T, x) = U(x)$, we have that the terminal condition for $v^{(1)}$ is: $v^{(1)}(T, x) = 0$.

We note from (20) that

$$D_2 v^{(0)} = -D_1 v^{(0)}, \tag{28}$$

so we could alternatively write (26) as

$$\mathcal{L}_{t,x}v^{(1)} = -\frac{1}{2}\rho_1 B D_1 D_2 v^{(0)}, \qquad v^{(1)}(T, x) = 0.$$

We also observe that (26) is a linear PDE for $v^{(1)}$, with varying coefficients that depend on $v^{(0)}$, and a source term also depending on $v^{(0)}$. In the next subsection, we derive an explicit expression for $v^{(1)}$ in terms of $v^{(0)}$.

2.3 Computation of the Value Function Correction

We first introduce three crucial lemmas that will lead to the solution of the linear PDE problem (26) with a zero terminal condition that will be given in Proposition 2.7.

2.3.1 Risk-Tolerance Equation and Commutation Result

The first lemma shows that, remarkably, the classical (constant λ) Merton risk-tolerance function, defined in Section 2.1, satisfies its own autonomous PDE.

Lemma 2.3. *The classical Merton risk-tolerance function $R(t, x; \lambda)$, defined in (11), satisfies the fast diffusion PDE:*

$$R_t + \frac{1}{2}\lambda^2 R^2 R_{xx} = 0. \tag{29}$$

Proof. Recall from Section 2.1 that R is smooth in (t, x). Differentiating (14) with respect to x gives

$$M_{tx} = \frac{1}{2}\lambda^2 R^2 M_{xxx} + \lambda^2 R R_x M_{xx}.$$

But from $RM_{xx} = -M_x$, we have $R^2 M_{xxx} = (R_x + 1)M_x$, and so

$$M_{tx} = \frac{1}{2}\lambda^2 (R_x + 1)M_x - \lambda^2 R_x M_x,$$

which gives that

$$M_{tx} + \frac{1}{2}\lambda^2 (R_x - 1)M_x = 0. \tag{30}$$

Next, differentiating (11) with respect to t gives

$$R_t = -\frac{M_{tx}}{M_{xx}} + \frac{M_x}{(M_{xx})^2}M_{txx}. \tag{31}$$

Differentiating (30) with respect to x, we have

$$M_{txx} = -\frac{1}{2}\lambda^2 (R_x - 1)M_{xx} - \frac{1}{2}\lambda^2 R_{xx} M_x,$$

and substituting this and (30) into (31) establishes (29). $\qquad\square$

Remark 2.4. The PDE (29) solved by the risk-tolerance function $R(t, x; \lambda)$ first appeared in Black [1968], and is also referred to as the fast diffusion equation in Musiela and Zariphopoulou [2010].

Next, we provide a commutation result that will allow for easy verification of the formula for $v^{(1)}$ to come in Proposition 2.7.

Lemma 2.5. *The operators $\mathcal{L}_{t,x}(\lambda)$ and D_1, defined in (13) and (12) respectively, acting on smooth functions of (t, x), commute:*

$$\mathcal{L}_{t,x}(\lambda)D_1 = D_1 \mathcal{L}_{t,x}(\lambda).$$

Proof. For any smooth $w(t, x)$, we compute

$$
\begin{aligned}
D_2 D_1 w - D_1 D_2 w &= R^2 \frac{\partial^2}{\partial x^2}(R w_x) - R\frac{\partial}{\partial x}(R^2 w_{xx}) \\
&= R^2 (R_{xx} w_x + 2R_x w_{xx} + R w_{xxx}) - R(2R R_x w_{xx} + R^2 w_{xxx}) \\
&= R^2 R_{xx} w_x.
\end{aligned} \tag{32}
$$

8

Then

$$\mathcal{L}_{t,x}(\lambda)D_1 w = \left(\frac{\partial}{\partial t} + \frac{1}{2}\lambda^2 D_2 + \lambda^2 D_1\right)D_1 w$$

$$= D_1\left(\frac{\partial}{\partial t} + \frac{1}{2}\lambda^2 D_2 + \lambda^2 D_1\right)w + \left(R_t + \frac{1}{2}\lambda^2 R^2 R_{xx}\right)w_x$$

$$= D_1\mathcal{L}_{t,x}(\lambda)w,$$

where we have used (29) and (32). $\qquad\square$

The third lemma shows that just as M satisfies the PDE (14), so also do its derivatives (or "Greeks") $D_1^k M$.

Lemma 2.6. *Recall the operators $\mathcal{L}_{t,x}(\lambda)$ and D_1, defined in (13) and (12) respectively, and the classical Merton value function M introduced in Section 2.1. We have that*

$$\mathcal{L}_{t,x}(\lambda)D_1^k M = 0, \qquad k = 1, 2, \cdots.$$

Proof. This follows from repeated use of Lemma 2.5 to establish $\mathcal{L}_{t,x}(\lambda)D_1^k = D_1^k \mathcal{L}_{t,x}(\lambda)$, and then using (14). $\qquad\square$

2.3.2 Explicit Expression for $v^{(1)}$

We can now give the main result of this subsection.

Proposition 2.7. *The linear PDE (26) with zero terminal condition $v^{(1)}(T, x) = 0$ has a unique solution given by*

$$v^{(1)}(t, x) = -(T - t)\frac{1}{2}\rho_1 B D_1^2 v^{(0)}(t, x), \tag{33}$$

or, equivalently,

$$v^{(1)}(t, x) = (T - t)\frac{1}{2}\rho_1 B D_1 D_2 v^{(0)}(t, x). \tag{34}$$

Proof. First we show that (33) is a solution of (26). We compute:

$$\mathcal{L}_{t,x}(\overline{\lambda})\left(-(T - t)\frac{1}{2}\rho_1 B D_1^2 v^{(0)}\right) = \frac{1}{2}\rho_1 B D_1^2 v^{(0)} - (T - t)\frac{1}{2}\rho_1 B\mathcal{L}_{t,x}(\overline{\lambda})D_1^2 v^{(0)} = \frac{1}{2}\rho_1 B D_1^2 v^{(0)},$$

because $\mathcal{L}_{t,x}(\overline{\lambda})D_1^2 v^{(0)} = 0$ by Lemma 2.6 and (19). Therefore the expression given in (33) solves the linear PDE (26). It also satisfies the zero terminal condition. We obtain (34) by using (28) in (33).

For the uniqueness, we work as follows. We define the new variable

$$\xi = -\log v_x^{(0)}(t, x) + \frac{1}{2}\overline{\lambda}^2(T - t), \tag{35}$$

which is a well-defined one-to-one transformation, as $v^{(0)}$ is strictly increasing and strictly concave. Making the substitution $v^{(0)}(t, x) = w^{(0)}(t', \xi)$, where $t' = t$, we have from (20):

$$\mathcal{L}_{t,x}(\overline{\lambda})v^{(0)} = \mathcal{H}(\overline{\lambda})w^{(0)} = \frac{\partial w^{(0)}}{\partial t'} + \frac{1}{2}\overline{\lambda}^2\frac{\partial^2 w^{(0)}}{\partial\xi^2}.$$

Therefore, $w^{(0)}$ solves the backwards heat equation $\mathcal{H}w^{(0)} = 0$, with terminal condition that depends on the solution through the ξ transformation. Applying the same change of variables to equation (26), we find that the correction is given by $v^{(1)}(t, x) = w^{(1)}(t', \xi)$, where $w^{(1)}$ solves the heat equation with source:

$$\mathcal{H}w^{(1)} = \frac{1}{2}\rho_1 B w_{\xi\xi}^{(0)}, \qquad w^{(1)}(T, \xi) = 0.$$

Uniqueness of the solution (33) follows from classical results for this equation. $\qquad\square$

The benefit of the formula (33) is that the correction $v^{(1)}$, which quantifies the principal effect of stochastic volatility when the correlation $\rho_1 \neq 0$, can be computed from derivatives of $v^{(0)}$. If $v^{(0)}$ is explicitly known, for instance with power utility then so is $v^{(1)}$, and we give the explicit formulas in that case for the full multiscale model in Section 6.1. Or, if $v^{(0)}$ is available numerically, then $v^{(1)}$ is obtained simply by numerical differencing (illustrated in Section 6.2 with mixture of power utilities).

We also remark that the transformation (35) can be written

$$\xi = \int_{\alpha(t)}^{x} \frac{1}{R(t, u; \overline{\lambda})}\, du + \frac{1}{2}\overline{\lambda}^2 (T - t),$$

where $\alpha(t)$ is defined by $v_x^{(0)}(t, \alpha(t)) = 1$.

2.4 Comparison with Option Pricing Asymptotics

We briefly review the fast mean-reverting option pricing asymptotic approximation described in Fouque et al. [2011], as there are some remarkable similarities one would not expect. The *no arbitrage* price of a European option with payoff $h(S_T)$ (under zero interest rates) is given by the following conditional expectation:

$$P^\varepsilon(t, S, y) = \boldsymbol{E}^\star \{h(S_T) \mid S_t = S, Y_t = y\},$$

under the (market-selected) risk-neutral measure $\boldsymbol{P}^\star$, where the dynamics of (S, Y) is described by

$$dS_t = \sigma(Y_t) S_t\, dW_t^{(0)\star},$$

$$dY_t = \left(\frac{1}{\varepsilon}b(Y_t) - \frac{1}{\sqrt{\varepsilon}}a(Y_t)\Lambda(Y_t)\right) dt + \frac{1}{\sqrt{\varepsilon}}a(Y_t)\, dW_t^{(1)\star}.$$

Here, $W^{(0)\star}$ and $W^{(1)\star}$ are $\boldsymbol{P}^\star$-Brownian motions with correlation structure $\boldsymbol{E}^\star\{dW^{(0)\star} dW^{(1)\star}\} = \rho_1\, dt$, and Λ is the market price of volatility risk. Then a singular perturbation analysis of the linear PDE problem that is solved by P^ε shows that

$$P^\varepsilon(t, S, y) = P_{BS}(t, S; \bar\sigma) + \sqrt{\varepsilon}\, P_1(t, S) + \cdots,$$

where the zeroth order term is the Black-Scholes option price with square averaged volatility $\bar\sigma^2 = \langle\sigma(\cdot)^2\rangle$. It does not depend on the current level $Y_t = y$, and it is the solution of the PDE problem

$$\mathcal{L}_{BS} P_{BS} = 0, \qquad P_{BS}(T, S) = h(S),$$

where the Black-Scholes operator is defined by

$$\mathcal{L}_{BS} = \frac{\partial}{\partial t} + \frac{1}{2}\bar\sigma^2 \mathcal{D}_2, \qquad \mathcal{D}_k = S^k \frac{\partial^k}{\partial S^k}, \quad k = 1, 2, \cdots.$$

Note that the operators $\mathcal{D}_k$ relevant for this problem are logarithmic derivatives that can be converted to a polynomial of regular derivatives by changing to log-stock variables.

The correction term P_1 also does not depend on the current level $Y_t = y$, and is the solution of the inhomogeneous Black-Scholes PDE

$$\mathcal{L}_{BS} P_1 = -(V_3 \mathcal{D}_1 \mathcal{D}_2 + V_2 \mathcal{D}_2)\, P_{BS}, \tag{36}$$

with zero terminal condition: $P_1(T, S) = 0$. The constant group parameters V_3 and V_2 contain the effect of the correlation ρ_1 and the volatility risk premium Λ respectively:

$$V_3 = -\frac{1}{2}\rho_1\langle a\sigma\phi'\rangle, \qquad V_2 = \frac{1}{2}\langle a\Lambda\phi'\rangle,$$

where $\phi(y)$ is a solution of the equation $\mathcal{L}_0\phi = \sigma(y)^2 - \bar{\sigma}^2$. Then, it is straightforward to show that the operators $\mathcal{L}_{BS}$ and $\mathcal{D}_k$ commute, and as a consequence, the explicit solution P_1 of (36) is given by

$$P_1 = (T - t)(V_3\mathcal{D}_1\mathcal{D}_2 + V_2\mathcal{D}_2)P_{BS}.$$

Comparing with the analysis of the nonlinear Merton problem in the previous sections: the role of the stock price variable S is played by the wealth variable x; the role of the Black-Scholes price P_{BS} is played by the Merton value function M; the role of the square-averaged volatility parameter $\bar{\sigma}$ by the square-averaged Sharpe ratio $\overline{\lambda}$; the role of the $\mathcal{D}_k$ by D_k; the role of the Black-Scholes operator $\mathcal{L}_{BS}$ by $\mathcal{L}_{t,x}$. Furthermore, *just as the option price correction can be found in terms of Greeks up to third-order ($\mathcal{D}_1\mathcal{D}_2$) in the stock price of P_{BS}, the correction $v^{(1)}$ to the value function can be found in terms of D_1D_2 derivatives of $v^{(0)}$ (formula (34)).* The parameter V_3 in the options problem is replaced by $\frac{1}{2}\rho_1 B$ in the portfolio problem (with θ in (24) playing the role of ϕ), and there is no market price of volatility risk in the control problem which is with respect to the historical measure $\boldsymbol{P}$.

We also point out that a similar comparison with option pricing asymptotics can be made with the slow scale volatility expansion we will construct in Section 3 as well as the combined multiscale expansion in Section 4, but we omit the comparison here for space. In the option pricing problem, model hypotheses and a proof of accuracy of the asymptotic approximation is given in [Fouque et al., 2011, Chapter 4]. In Section 6.3, we provide a proof of accuracy for the Merton problem with power utility and one factor of stochastic volatility (either fast or slow), when the problem can be transformed to a linear equation.

2.5 Optimal Portfolio

We now analyze and interpret how the principal expansion terms $v^{(0)}$ and $v^{(1)}$ for the value function can be used in the expression for the optimal portfolio π^* in (7), which leads to an approximate feedback policy of the form

$$\pi^*(t, x, y) = \pi^{(0)}(t, x, y) + \sqrt{\varepsilon}\,\pi^{(1)}(t, x, y) + \cdots.$$

2.5.1 Zeroth Order Strategy

First, we insert the zeroth order term $v^{(0)}$ in the expansion for V^ε, which we found in Section 2.2.1. This gives the zeroth order optimal portfolio function in feedback form

$$\pi^{(0)}(t, x, y) := -\frac{\lambda(y)}{\sigma(y)}\frac{v_x^{(0)}}{v_{xx}^{(0)}} = \frac{\lambda(y)}{\sigma(y)}R(t, x; \overline{\lambda}).$$

Now, in the case of the Merton problem with *constant* Sharpe ratio λ_c and constant volatility σ_c, the optimal strategy is given by

$$\pi^{(M)}(t, x) := \frac{\lambda_c}{\sigma_c}R(t, x; \lambda_c).$$

The *naive Merton strategy* would be to use the strategy $\pi^{(M)}$, even when the coefficients (μ, σ) are driven by stochastic factors. The question would be which constant coefficients (λ_c, σ_c) to use? In the context of our multiscale stochastic volatility models, this question can be answered by finding the zeroth order of the optimal strategy among strategies that do not depend on the fast factor. This will be explained in more detail in Section 5.

The *moving Merton strategy* consists in using $\pi^{(M)}$, but with coefficients following the varying factors. In the case of only the fast factor Y_t, it is given by

$$\pi^{(MM)}(t, x, y) := \frac{\lambda(y)}{\sigma(y)}R(t, x; \lambda(y)).$$

11

Note that in the case of power or log utility, $\pi^{(0)}$ and $\pi^{(MM)}$ coincide as R does not depend on λ, and they are myopic as R does not depend on time.

The asymptotics identifies that, for general utility functions, $\pi^{(0)}$ is neither a naive Merton strategy $\pi^{(M)}$ (with some averaged coefficients), nor the moving Merton strategy $\pi^{(MM)}$, but rather a hybrid in which the pre-factor λ/σ moves with the stochastic factor Y, while the risk tolerance component R is computed using the constant averaged Sharpe ratio $\overline{\lambda}$ defined in (17).

Moreover, we demonstrate in Appendix A that using our zeroth order suboptimal strategy $\pi^{(0)}$ results in the optimal value, not only up to the principal term $v^{(0)}$, but also up to first order $\sqrt{\varepsilon}$ correction $v^{(0)} + \sqrt{\varepsilon}\, v^{(1)}$.

2.5.2   First Order Correction to Optimal Portfolio

Our approximation to the optimal strategy can be made more accurate by going to the next order terms. Inserting the expansion of the value function up to terms in $\sqrt{\varepsilon}$ in (7) gives $\pi^* = \widetilde{\pi^{\varepsilon}} + \textit{higher order terms}$, where we define

$$
\widetilde{\pi^{\varepsilon}} = -\frac{\lambda(y)}{\sigma(y)}\frac{(v_x^{(0)} + \sqrt{\varepsilon}\, v_x^{(1)})}{(v_{xx}^{(0)} + \sqrt{\varepsilon}\, v_{xx}^{(1)})} - \sqrt{\varepsilon}\frac{\rho_1 a(y)}{\sigma(y)}\frac{v_{xy}^{(2)}}{v_{xx}^{(0)}}
$$

$$
= \pi^{(0)} - \sqrt{\varepsilon}\left\{\frac{\lambda(y)}{\sigma(y)}\left[\frac{v_x^{(1)}}{v_{xx}^{(0)}} - \frac{v_x^{(0)}}{(v_{xx}^{(0)})^2}v_{xx}^{(1)}\right] + \frac{1}{2}\frac{\rho_1 a(y)}{\sigma(y)v_{xx}^{(0)}}\theta'(y)\frac{\partial}{\partial x}\left(\frac{(v_x^{(0)})^2}{v_{xx}^{(0)}}\right)\right\},
$$

and we used formula (23) for $v^{(2)}$. Then, using the definition of D_1 and D_2 in (20), and substituting for $v^{(1)}$ from (34) gives

$$
\widetilde{\pi^{\varepsilon}} = \pi^{(0)} + \frac{\sqrt{\varepsilon}\,\rho_1}{2\sigma(y)v_x^{(0)}}\left(B\lambda(y)(T-t)(D_1+D_2) - a(y)\theta'(y)\,\mathbb{I}\right)D_1 D_2 v^{(0)}, \tag{37}
$$

where the constant B was defined in (27), and $\mathbb{I}$ denotes the identity operator. Explicit formulas for power utilities and for the full multiscale model are given in Section 6.1.

We will see in formula (48) that $D_2 v^{(0)}$ can be written proportional to the sensitivity of $v^{(0)}$ with respect to the Sharpe ratio, which fluctuates with the stochastic factor Y. Therefore the terms multiplying $\sqrt{\varepsilon}$ in (37) can be thought of as the principal terms hedging this factor risk. We will comment more on this in Section 4.2 for the full multiscale model.

3   Slow Scale Volatility Asymptotics

We now perform an asymptotic analysis under the assumption that stochastic volatility is slowly fluctuating. We show in Section 4 that under two-factor multiscale stochastic volatility models, with both a fast and a slow factor, how the results of the fast analysis in the previous section and the slow analysis in this section combine.

We have the model

$$
dS_t = \mu(Z_t)S_t\, dt + \sigma(Z_t)S_t\, dW_t^{(0)}
$$

$$
dZ_t = \delta c(Z_t)\, dt + \sqrt{\delta}g(Z_t)\, dW_t^{(2)}, \tag{38}
$$

where $W^{(0)}$ and $W^{(2)}$ are Brownian motions with instantaneous correlation coefficient between volatility and stock price shocks $\rho_2 \in (-1, 1)$, and δ is the small time-scale parameter for expansion. Here, we assume that $Z_t = Z_{\delta t}^{(1)}$ in distribution, where $Z^{(1)}$ is a diffusion process with drift and diffusion coefficients c and g respectively, which explains the $\sqrt{\delta}$ in front of the Brownian motion in (38). We do not need any ergodicity

assumptions on $Z^{(1)}$ for the slow scale asymptotics in the limit $\delta \downarrow 0$, but we require the coefficients $\mu(z)$ and $\sigma(z)$ to be differentiable.

The HJB PDE problem for the value function of the Merton problem

$$V^\delta(t, x, z) = \sup_\pi \mathbf{E}\left\{U(X_T) \mid X_t = x, Z_t = z\right\},$$

is

$$V_t^\delta + \delta \mathcal{M}_2 V^\delta - \frac{\left(\lambda(z)V_x^\delta + \sqrt{\delta}\rho_2 g(z)V_{xz}^\delta\right)^2}{2V_{xx}^\delta} = 0, \qquad V^\delta(T, x, z) = U(x), \tag{39}$$

where $\mathcal{M}_2$ is the infinitesimal generator of the process $Z^{(1)}$:

$$\mathcal{M}_2 = \frac{1}{2}g(z)^2\frac{\partial^2}{\partial z^2} + c(z)\frac{\partial}{\partial z}, \tag{40}$$

and the Sharpe ratio $\lambda(z) = \frac{\mu(z)}{\sigma(z)}$.

Assumption 3.1. The value function $V^\delta(t, x, z)$ is smooth on $[0, T] \times R_+ \times R$, and is strictly increasing and strictly concave in the wealth argument x for each $z \in R$ and $t \in [0, T)$. Moreover, it is the unique solution in this class of the HJB PDE problem (39).

3.1 Slow Scale Expansion

We look for an expansion for the value function V^δ of the form

$$V^\delta(t, x, z) = v^{(0)}(t, x, z) + \sqrt{\delta}\,v^{(1)}(t, x, z) + \delta v^{(2)}(t, x, z) + \delta^{3/2}v^{(3)}(t, x, z)\cdots. \tag{41}$$

Then it follows by setting $\delta = 0$ in (39) that $v^{(0)}$ solves

$$v_t^{(0)} - \frac{1}{2}\lambda(z)^2\frac{(v_x^{(0)})^2}{v_{xx}^{(0)}} = 0, \qquad v^{(0)}(T, x, z) = U(x). \tag{42}$$

Therefore, comparing with (10), the principal term is the Merton value function with the current Sharpe ratio $\lambda(z) = \frac{\mu(z)}{\sigma(z)}$:

$$v^{(0)}(t, x, z) = M(t, x; \lambda(z)), \tag{43}$$

where $M(t, x; \lambda)$ was defined in Section 2.1. As with the fast factor zeroth order approximation to the value function given in (19), the zeroth order approximation in the slow factor model is the constant parameter Merton value function, but with $\lambda(z)$, the current Sharpe ratio, instead of the averaged quantity $\overline{\lambda}$.

We recall notation introduced in Section 2.1, adapted here for the slow scale analysis:

$$\mathcal{L}_{t,x}(\lambda(z)) = \frac{\partial}{\partial t} + \frac{1}{2}\lambda(z)^2 D_2 + \lambda(z)^2 D_1, \tag{44}$$

$$D_k = R(t, x; \lambda(z))^k\frac{\partial^k}{\partial x^k}, \qquad k = 1, 2, \cdots, \tag{45}$$

$$R(t, x; \lambda(z)) = -\frac{v_x^{(0)}(t, x, z)}{v_{xx}^{(0)}(t, x, z)}. \tag{46}$$

With this notation, we can re-write (42) as

$$\mathcal{L}_{t,x}(\lambda(z))v^{(0)} = 0, \qquad v^{(0)}(T, x, z) = U(x).$$

Taking the order $\sqrt{\delta}$ terms after inserting the expansion (41) into the PDE (39) leads to

$$\mathcal{L}_{t,x}(\lambda(z))v^{(1)} = \rho_2\lambda(z)g(z)\left(\frac{v_x^{(0)}v_{xz}^{(0)}}{v_{xx}^{(0)}}\right), \qquad v^{(1)}(T, x, z) = 0. \tag{47}$$

13

3.1.1 "Vega-Gamma" Relation

The following Lemma enables us to construct the solution to (47).

Lemma 3.2. *The Merton value function $M(t, x; \lambda)$ introduced in Section 2.1 satisfies the "Vega-Gamma" relation*

$$\frac{\partial M}{\partial \lambda} = -(T - t)\lambda R^2 \frac{\partial^2 M}{\partial x^2}, \tag{48}$$

where R denotes the risk-tolerance function $R = -M_x/M_{xx}$.

Proof. Recall the notation $\mathcal{L}_{t,x}$ and D_k introduced in (13) and (12). We have that $\mathcal{L}_{t,x}(\lambda)M = 0$. Differentiating this PDE with respect to λ gives

$$
\begin{aligned}
\mathcal{L}_{t,x}(\lambda)M_\lambda &= -\frac{1}{2}\left[\frac{\partial}{\partial \lambda}(\lambda R)^2\right]M_{xx} - \left[\frac{\partial}{\partial \lambda}(\lambda^2 R)\right]M_x \\
&= -\lambda D_2 M - 2\lambda D_1 M - [RM_{xx} + M_x]\lambda^2 R_\lambda \\
&= -\lambda D_2 M - 2\lambda D_1 M = \lambda D_2 M.
\end{aligned} \tag{49}
$$

By differentiating the terminal condition $M(T, x; \lambda) = U(x)$ with respect to λ, we have $M_\lambda(T, x; \lambda) = 0$. Using Lemma 2.5, we see that the solution to the PDE (49) with zero terminal condition is given by $M_\lambda = -(T - t)\lambda D_2 M$, which gives (48). □

Expression (48) is similar to the Vega-Gamma relationship for European option prices in the Black-Scholes model (see, for instance [Fouque et al., 2011, Section 1.3.5]), which says the following. The Black-Scholes European option price when volatility is a constant σ, $P_{BS}(t, S; \sigma)$ in the notation of Section 2.4, satisfies

$$\frac{\partial}{\partial \sigma}P_{BS} = (T - t)\sigma S^2 \frac{\partial^2}{\partial S^2}P_{BS}.$$

This relationship is used to connect convex payoffs (positive Gamma) to long volatility positions (positive Vega). The signs of the terms on each side of (48) are consistent because the "Vega" on the left is positive (value increases with Sharpe ratio) and the "Gamma" on the right is negative because M is concave in x.

3.1.2 Explicit Expression for Slow Scale Value Function Correction

We can now give an explicit expression for $v^{(1)}$ in terms of $v^{(0)}$ using the previous Lemma 3.2, from which we obtain

$$v_z^{(0)} = -(T - t)\lambda\lambda' D_2 v^{(0)}. \tag{50}$$

We first re-write the equation (47) for $v^{(1)}(t, x, z)$ as:

$$\mathcal{L}_{t,x}(\lambda(z))v^{(1)} = -\rho_2\lambda(z)g(z)D_1 v_z^{(0)}. \tag{51}$$

Proposition 3.3. The slow scale correction is given by

$$v^{(1)}(t, x, z) = \frac{1}{2}(T - t)\rho_2\lambda(z)g(z)D_1 v_z^{(0)}. \tag{52}$$

Proof. Using the "Vega-Gamma" relation (50) to convert the $v_z^{(0)}$ term in (51) to a $D_2 v^{(0)}$, we see that $v^{(1)}$ solves

$$\mathcal{L}_{t,x}(\lambda(z))v^{(1)} = -\rho_2\lambda(z)g(z)D_1\left(-(T - t)\lambda(z)\lambda'(z)D_2 v^{(0)}\right) = -(T - t)\rho_2\lambda^2(z)g(z)\lambda'(z)D_1^2 v^{(0)}.$$

Now, similar to Proposition 2.7, we see the solution with zero terminal condition is

$$v^{(1)} = \frac{1}{2}(T-t)^2 \rho_2 \lambda^2 g \lambda' D_1^2 v^{(0)} = -\frac{1}{2}(T-t)^2 \rho_2 \lambda^2 g \lambda' D_1 D_2 v^{(0)}, \tag{53}$$

because $\mathcal{L}_{t,x} D_1^2 v^{(0)} = 0$ by Lemma 2.6. Finally, we use the "Vega-Gamma" relation (50) again to convert the $D_2 v^{(0)}$ term in (53) back into a $v_z^{(0)}$ term to obtain (52). $\qquad\square$

The benefit of the formula (52) is that the correction $v^{(1)}$, which quantifies the principal effect of stochastic volatility when the correlation $\rho_2 \neq 0$, can be computed from derivatives of $v^{(0)}$. In the slow scale case the appropriate derivative is $D_1 v_z^{(0)}$. The expression in (53) appears, up to the pre-factors, very similar to the formula (34) for the correction in the fast asymptotics, except for the additional $(T-t)$ factor. This is quite intuitive: in terms of the value function, closer to maturity the slow volatility factor is less important than the fast factor.

3.2 Optimal Portfolio

The optimal strategy in feedback form is given by

$$\pi^*(t, x, z) = -\frac{\lambda(z)}{\sigma(z)} \frac{V_x^\delta}{V_{xx}^\delta} - \sqrt{\delta} \frac{\rho_2 g(z)}{\sigma(z)} \frac{V_{xz}^\delta}{V_{xx}^\delta}. \tag{54}$$

Inserting the expressions for $v^{(0)}$ and $v^{(1)}$ in the expansion for V^δ into (54) gives an approximation for π^*.

3.2.1 Zeroth Order Strategy

First we insert the zeroth order term $v^{(0)}$ in the expansion for V^δ, which we found in (42). This gives the zeroth order strategy

$$\pi^{(0)}(t, x, z) := \frac{\lambda(z)}{\sigma(z)} R(t, x; \lambda(z)).$$

In this case $\pi^{(0)}$ is the moving Merton strategy $\pi^{(0)} = \pi^{(MM)}$, meaning as we discussed in Section 2.5.1, it is the Merton strategy updated with the moving level z of the factor process Z.

We demonstrate in Appendix B that using our zeroth order suboptimal strategy $\pi^{(0)}$ results in the optimal value not only up to the principal term $v^{(0)}$, but also up to first order $\sqrt{\delta}$ correction $v^{(0)} + \sqrt{\delta} v^{(1)}$. This is in line with the findings of Chacko and Viceira [2005], who find the intertemporal hedging terms in their model and optimization problem are relatively small.

3.2.2 First Order Correction to Optimal Portfolio

Our approximation to the optimal strategy can be made more accurate by going to the next order terms. Inserting the expansion for the value function V^δ up to the $\sqrt{\delta}$ terms into (54) gives $\pi^* = \widetilde{\pi}^\delta + higher\ order\ terms$, where we define

$$\widetilde{\pi}^\delta = \frac{\lambda(z)}{\sigma(z)} R(t, x; \lambda(z)) + \frac{\sqrt{\delta} \rho_2 g(z)}{\sigma(z) v_x^{(0)}} \left(\frac{1}{2}(T-t)\lambda^2(z)(D_2 + D_1) + \mathbb{I} \right) D_1 v_z^{(0)}. \tag{55}$$

The terms multiplying $\sqrt{\delta}$ in (55) can be thought of as the principal terms hedging the risk from the factor Z. We will comment more on this in Section 4.2 for the full multiscale model.

4 Merton Problem under Multiscale Stochastic Volatility

We return to the two-factor multiscale stochastic volatility model (1), introduced in Section 1, where there is one fast volatility factor, and one slow. We show that the separate fast and slow expansions to first order (Sections 2 and 3) essentially combine, but with some modification of the averaged parameters involved.

Under our simplifying assumption of zero interest rates, the wealth process X of an investor holding π_t dollars in the stock at time t follows

$$dX_t = \pi_t \mu(Y_t, Z_t)\, dt + \pi_t \sigma(Y_t, Z_t)\, dW_t^{(0)}.$$

The value function

$$V^{\varepsilon,\delta}(t, x, y, z) = \sup_{\pi} \boldsymbol{E}\left\{U(X_T) \mid X_t = x, Y_t = y, Z_t = z\right\}$$

has the associated HJB PDE problem

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3 + \delta\mathcal{M}_2 + \frac{\partial}{\partial t}\right)V^{\varepsilon,\delta} + \mathrm{NL}^{\varepsilon,\delta} = 0, \qquad V^{\varepsilon,\delta}(T, x, y, z) = U(x), \tag{56}$$

where $\mathcal{L}_0$ was defined in (6), $\mathcal{M}_2$ in (40), and the linear operator $\mathcal{M}_3$ comes from the correlation between the Brownian motions driving the fast and slow factors:

$$\mathcal{M}_3 = \rho_{12}a(y)g(z)\frac{\partial^2}{\partial y\partial z}.$$

The nonlinear term is given by

$$\begin{aligned}
\mathrm{NL}^{\varepsilon,\delta} &= \max_{\pi}\left(\frac{1}{2}\pi^2\sigma^2 V_{xx}^{\varepsilon,\delta} + \pi\left[\mu V_x^{\varepsilon,\delta} + \frac{1}{\sqrt{\varepsilon}}\rho_1 a\sigma V_{xy}^{\varepsilon,\delta} + \sqrt{\delta}\rho_2 g\sigma V_{xz}^{\varepsilon,\delta}\right]\right) \\
&= -\frac{\left(\lambda V_x^{\varepsilon,\delta} + \frac{1}{\sqrt{\varepsilon}}\rho_1 a V_{xy}^{\varepsilon,\delta} + \sqrt{\delta}\rho_2 g V_{xz}^{\varepsilon,\delta}\right)^2}{2V_{xx}^{\varepsilon,\delta}},
\end{aligned}$$

where the Sharpe ratio is

$$\lambda(y, z) = \frac{\mu(y, z)}{\sigma(y, z)}. \tag{57}$$

The optimal strategy in feedback form is given by:

$$\pi^*(t, x, y, z) = -\frac{\left(\lambda V_x^{\varepsilon,\delta} + \frac{1}{\sqrt{\varepsilon}}\rho_1 a V_{xy}^{\varepsilon,\delta} + \sqrt{\delta}\rho_2 g V_{xz}^{\varepsilon,\delta}\right)}{\sigma V_{xx}^{\varepsilon,\delta}}. \tag{58}$$

Assumption 4.1. The value function $V^{\varepsilon,\delta}(t, x, y, z)$ is smooth on $[0, T]\times R_+\times R\times R$, and is strictly increasing and strictly concave in the wealth argument x for each $(y, z) \in R^2$ and $t \in [0, T)$. Moreover, it is the unique solution in this class of the HJB PDE problem (56).

4.1 Combined Expansion in Slow and Fast Scales

First we construct an expansion in powers of $\sqrt{\delta}$:

$$V^{\varepsilon,\delta} = V^{\varepsilon,0} + \sqrt{\delta}V^{\varepsilon,1} + \cdots, \tag{59}$$

so that $V^{\varepsilon,0}$ is obtained by setting $\delta = 0$ in the equation for $V^{\varepsilon,\delta}$:

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{\partial}{\partial t}\right)V^{\varepsilon,0} - \frac{\left(\lambda(y,z)V_x^{\varepsilon,0} + \frac{\rho_1 a(y)}{\sqrt{\varepsilon}}V_{xy}^{\varepsilon,0}\right)^2}{2V_{xx}^{\varepsilon,0}} = 0, \tag{60}$$

with terminal condition $V^{\varepsilon,0}(T, x, y, z) = U(x)$. This is the same HJB problem (8) as for the value function V^{ε} except that the Sharpe ratio depends on the current level z of the slow volatility factor, which enters as a parameter in the PDE (60). It is clear then that when we construct an expansion of $V^{\varepsilon,0}$ in powers of $\sqrt{\varepsilon}$:

$$V^{\varepsilon,0} = v^{(0)} + \sqrt{\varepsilon}\, v^{(1,0)} + \cdots,$$

we will obtain, as in Section 2, that $v^{(0)}(t, x, z)$ is the Merton value function with constant Sharpe ratio $\overline{\lambda}(z)$:

$$v^{(0)}(t, x, z) = M(t, x; \overline{\lambda}(z)), \tag{61}$$

where $\overline{\lambda}^2(z) = \langle \lambda^2(\cdot, z)\rangle$. That is, the Sharpe ratio is square-averaged over the fast factor with respect to its invariant distribution, and evaluated at the current level of the slow factor.

The appropriate modifications for the risk-tolerance functions and our usual operators are:

$$R(t, x; \overline{\lambda}(z)) = -\frac{v_x^{(0)}(t, x, z)}{v_{xx}^{(0)}(t, x, z)}, \qquad D_k = R(t, x; \overline{\lambda}(z))^k \frac{\partial^k}{\partial x^k}, \qquad \mathcal{L}_{t,x}(\overline{\lambda}(z)) = \frac{\partial}{\partial t} + \frac{1}{2}\overline{\lambda}(z)^2 D_2 + \overline{\lambda}(z)^2 D_1,$$

and so $v^{(0)}(t, x, z)$ solves

$$\mathcal{L}_{t,x}(\overline{\lambda}(z))v^{(0)} = 0, \qquad v^{(0)}(T, x, z) = U(x).$$

Following Proposition 2.7, the correction term $v^{(1,0)}$ is given by

$$v^{(1,0)}(t, x, z) = (T - t)\frac{1}{2}\rho_1 B(z) D_1 D_2 v^{(0)}(t, x, z),$$

where

$$B(z) = \left\langle \lambda(\cdot, z)a(\cdot)\frac{\partial\theta}{\partial y}(\cdot, z)\right\rangle, \tag{62}$$

and $\theta(y, z)$ is a solution of the ODE (in y)

$$\mathcal{L}_0\theta = \lambda^2(y, z) - \overline{\lambda}^2(z). \tag{63}$$

Next we return to the slow scale expansion (59) and extract the order $\sqrt{\delta}$ terms in (56) to obtain the following equation for $V^{\varepsilon,1}$:

$$\frac{1}{\varepsilon}\mathcal{L}_0 V^{\varepsilon,1} + \frac{1}{\sqrt{\varepsilon}}\mathcal{M}_3 V^{\varepsilon,0} + \frac{\partial}{\partial t}V^{\varepsilon,1} + \mathrm{NL}^{(1)} = 0, \tag{64}$$

where

$$\mathrm{NL}^{(1)} = -\frac{1}{V_{xx}^{\varepsilon,0}}\left(\lambda V_x^{\varepsilon,0} + \frac{\rho_1 a}{\sqrt{\varepsilon}}V_{xy}^{\varepsilon,0}\right)\left(\rho_2 g V_{xz}^{\varepsilon,0} + \lambda V_x^{\varepsilon,1} + \frac{\rho_1 a}{\sqrt{\varepsilon}}V_{xy}^{\varepsilon,1}\right) + \frac{1}{2}\left(\frac{\lambda V_x^{\varepsilon,0} + \frac{\rho_1 a}{\sqrt{\varepsilon}}V_{xy}^{\varepsilon,0}}{V_{xx}^{\varepsilon,0}}\right)^2 V_{xx}^{\varepsilon,1}.$$

The terminal condition is $V^{\varepsilon,1}(T, x, y, z) = 0$.

We look for an expansion

$$V^{\varepsilon,1} = v^{(0,1)} + \sqrt{\varepsilon}\,v^{(1,1)} + \varepsilon\,v^{(2,1)} + \cdots,$$

where we are only interested here in the first term which will give the principal slow scale correction to the value function. We observe that as the first two terms in $V^{\varepsilon,0}$ do not depend on y and $\mathcal{M}_3$ takes a derivative in y, then the term $\varepsilon^{-1/2}\mathcal{M}_3 V^{\varepsilon,0}$ in the PDE (64) is of order $\sqrt{\varepsilon}$. Similarly, the terms in $\mathrm{NL}^{(1)}$ involving $V^{\varepsilon,0}_{xy}$ are also order $\sqrt{\varepsilon}$ and will not play a role in finding $v^{(0,1)}$.

The order ε^{-1} terms in (64) give $\mathcal{L}_0 v^{(0,1)} = 0$ and we take $v^{(0,1)} = v^{(0,1)}(t,x,z)$, independent of y. At order $\varepsilon^{-1/2}$, we have $\mathcal{L}_0 v^{(1,1)} = 0$ and so again $v^{(1,1)} = v^{(1,1)}(t,x,z)$. At order one:

$$\mathcal{L}_0 v^{(2,1)} + v^{(0,1)}_t - \rho_2 g \lambda \frac{v^{(0)}_{xz} v^{(0)}_x}{v^{(0)}_{xx}} + \frac{1}{2}\lambda^2 \frac{(v^{(0)}_x)^2}{(v^{(0)}_{xx})^2} v^{(0,1)}_{xx} - \lambda^2 \frac{v^{(0)}_x}{v^{(0)}_{xx}} v^{(0,1)}_x = 0.$$

Viewed as a Poisson equation for $v^{(2,1)}$, this yields the following solvability condition for $v^{(0,1)}$:

$$\mathcal{L}_{t,x}(\overline{\lambda}(z)) v^{(0,1)} = \rho_2 \widehat{\lambda}(z) g(z) \frac{v^{(0)}_{xz} v^{(0)}_x}{v^{(0)}_{xx}},$$

where $\widehat{\lambda}(z) = \langle \lambda(\cdot, z)\rangle$. With zero terminal condition, this is the same PDE problem (47) as for the slow scale correction in Section 3, except with $\lambda(z)$ on the right side replaced by $\widehat{\lambda}(z)$. This change in constant does not affect the argument of Proposition 3.3, and so we conclude that

$$v^{(0,1)}(t,x,z) = \frac{1}{2}(T-t)\rho_2\widehat{\lambda}(z)g(z)D_1 v^{(0)}_z = -\frac{1}{2}(T-t)^2\rho_2\widehat{\lambda}(z)\overline{\lambda}(z)\overline{\lambda}'(z)g(z)D_1 D_2 v^{(0)},$$

where in the second expression we have used Lemma 3.2 to convert the $v^{(0)}_z$ Vega term to a $D_2 v^{(0)}$ Gamma term.

In summary, the first-order multiscale correction is given by $V^{\varepsilon,\delta}(t,x,y,z) = \widetilde{V^{\varepsilon,\delta}}(t,x,z) + \textit{higher order terms}$, where

$$\widetilde{V^{\varepsilon,\delta}}(t,x,z) := v^{(0)}(t,x,z) + \left(\frac{1}{2}\rho_1\sqrt{\varepsilon}(T-t)B(z) - \frac{1}{2}\sqrt{\delta}(T-t)^2\rho_2\widehat{\lambda}(z)\overline{\lambda}(z)\overline{\lambda}'(z)g(z)\right)D_1 D_2 v^{(0)}(t,x,z), \quad (65)$$

which depends on the square-averaged Sharpe ratio $\overline{\lambda}(z)$ as well as the straight average $\widehat{\lambda}(z)$, and the group parameter $B(z)$ defined in (62). This expression highlights that the principal stochastic volatility corrections to the Merton value function are proportional to the correlations between volatility factors and returns shocks (measured by ρ_1 and ρ_2). It also shows that given a numerical solution $v^{(0)}$, which solves a PDE in (t,x) only, with z as a parameter, the correction terms are obtained simply by computing the derivative $D_1 D_2 v^{(0)}$.

4.2 Multiscale Optimal Portfolio

The optimal portfolio up to orders $\sqrt{\varepsilon}$ and $\sqrt{\delta}$ for the multiscale model is obtained by inserting the value function approximation (65) into the optimal strategy feedback function (58), which leads to $\pi^* = \widetilde{\pi^{\varepsilon,\delta}} + \textit{higher order terms}$, where

$$\widetilde{\pi^{\varepsilon,\delta}} = \frac{\lambda(y,z)}{\sigma(y,z)}R(t,x;\overline{\lambda}(z)) + \frac{\sqrt{\varepsilon}\rho_1}{2\sigma(y,z)v^{(0)}_x}\left(B(z)\lambda(y,z)(T-t)(D_1+D_2) - a(y)\theta_y(y,z)\mathbb{I}\right)D_1 D_2 v^{(0)}$$

$$+ \frac{\sqrt{\delta}\rho_2 g(z)}{\sigma(y,z)v^{(0)}_x}\left(\frac{1}{2}(T-t)\lambda(y,z)\widehat{\lambda}(z)(D_2+D_1) + \mathbb{I}\right)D_1 v^{(0)}_z. \quad (66)$$

18

Here $v^{(0)}(t, x, z)$ is the Merton value function in (61), and $R(t, x; \overline{\lambda}(z))$ is its corresponding risk-tolerance function. The coefficient $\sigma(y, z)$ is the volatility function in (1), $\lambda(y, z)$ is the multifactor Sharpe ratio in (57), $\theta(y, z)$ is defined in (63), and the formula for $B(z)$ is given in (62).

The principal (zero order) strategy

$$\pi^{(0)}(t, x, y, z) := \frac{\lambda(y, z)}{\sigma(y, z)} R(t, x; \overline{\lambda}(z)) \tag{67}$$

is a moving Merton strategy with respect to the slow factor Z, as in the slow-only case (Section 3.2.1). In terms of the fast factor Y, it is a hybrid of naive Merton in the R component where the fast factor is averaged out, and moving Merton in the pre-factor λ/σ, as we found in the fast-only case in Section 2.5.1. Furthermore, combining the results in Appendices A and B, one finds that using $\pi^{(0)}$ in (67) recovers the optimal value function up to orders $\sqrt{\varepsilon}$ and $\sqrt{\delta}$. Note that this strategy requires tracking both fast and slow factors.

The formula (66) for the approximate optimal portfolio up to orders $\sqrt{\varepsilon}$ and $\sqrt{\delta}$ highlights the contribution from the volatility factor-returns correlations. It can also be interpreted as an expression in terms of the Merton strategy and the sensitivities (or "Greeks") of the Merton value function with respect to the wealth x and slow volatility factor level z. The specific Greeks involved are identified by the asymptotic analysis. These terms then comprise the principal parts of the hedging terms against volatility risk.

5   Practical Strategy

Tracking the fast factor Y requires high-frequency data and dealing with microstructure issues requiring sophisticated econometric techniques (see, among others, the book by Aït-Sahalia and Jacod [2014] for a detailed analysis of the difficulties). Many investors will not tackle these issues and will look for a practical (or lazy) strategy whose principal terms do not depend on tracking the fast moving volatility factor Y, but they do track the slow factor Z. In addition, ignoring the fast factor will likely lead to strategies that require less frequent rebalancing. For the multiscale stochastic volatility model (1), we propose such a strategy and quantify its loss of utility.

Naively, an investor ignoring the fast factor would use a one (slow) factor model of the form

$$dS_t = \bar{\mu}(Z_t) S_t \, dt + \bar{\sigma}(Z_t) S_t \, dW_t^{(0)},$$

with Z as in (1), and then use a moving Merton strategy of the (feedback) form

$$\bar{\pi}^{(0)}(t, x, z) = \frac{\bar{\mu}(z)}{\bar{\sigma}(z)^2} R\left(t, x; \frac{\bar{\mu}(z)}{\bar{\sigma}(z)}\right). \tag{68}$$

The key question is how should the coefficients $\bar{\mu}(z)$ and $\bar{\sigma}(z)$ relate to the full model coefficients $\mu(y, z)$ and $\sigma(y, z)$ in (1) to account for the presence of the fast volatility factor.

In Appendix C we derive the leading order term in the small (ε, δ) regime of the optimal Y-independent strategy. We find it is of the form (68), where the appropriate coefficients are

$$\bar{\sigma}(z)^2 = \langle \sigma(\cdot, z)^2 \rangle, \qquad \bar{\mu}(z) = \langle \mu(\cdot, z) \rangle,$$

the averages of the growth rate function $\mu(y, z)$ and the squared volatility $\sigma^2(y, z)$ with respect to the invariant distribution of Y. We also derive the leading order term $\bar{v}^{(0)}$ of the value function of following this strategy, and we now use it to quantify the suboptimality of the practical strategy.

From (68), we observe that the practical strategy is the constant Sharpe ratio Merton strategy using the fast-scale averaged growth rate and volatility parameters $\bar{\mu}(z)$ and $\bar{\sigma}(z)$ respectively, evaluated at the current

level of the slow factor. Indeed, it may happen that for some level z of the slow factor, $\bar{\mu}(z) = 0$, which implies the practical strategy is to hold no stock at that instant, even while the current Sharpe ratio may be positive, thus giving up some utility from the risky asset. To assess the suboptimality, we compare with the value function approximation derived in Section 4 for the multiscale model, when the fast factor is assumed observable, and the principal term $v^{(0)}(t, x, z)$ is given by (61). As $\bar{v}^{(0)}(t, x, z) = M(t, x; \frac{\bar{\mu}(z)}{\bar{\sigma}(z)})$, the suboptimality of the practical strategy is, to principal order,

$$v^{(0)}(t, x, z) - \bar{v}^{(0)}(t, x, z) = M(t, x; \overline{\lambda}(z)) - M\left(t, x; \frac{\bar{\mu}(z)}{\bar{\sigma}(z)}\right),$$

and is governed by the Cauchy-Schwarz gap

$$\overline{\lambda}^2 = \left\langle \frac{\mu^2}{\sigma^2} \right\rangle \geq \frac{\langle \mu^2 \rangle}{\bar{\sigma}^2} \geq \frac{\bar{\mu}^2}{\bar{\sigma}^2}.$$

In other words, as $\varepsilon \downarrow 0$, using the optimal strategy (observing both the fast factor Y and the slow factor Z) the value function converges to the Merton value with Sharpe ratio $\overline{\lambda}(z)$, while using the practical strategy, the expected utility converges to the smaller Merton value with Sharpe ratio $\frac{\bar{\mu}(z)}{\bar{\sigma}(z)}$.

In the case that μ is constant, we have that $\overline{\lambda}^2(z) = \mu^2/\sigma_\star^2(z)$, where $\sigma_\star(z)$ is the harmonically square-averaged volatility defined by

$$\frac{1}{\sigma_\star^2(z)} = \left\langle \frac{1}{\sigma^2(\cdot, z)} \right\rangle.$$

Then the limit value function $v^{(0)}$ is the Merton value as if the volatility was the averaged quantity $\sigma_\star(z)$, whereas the limit value of the practical strategy is the Merton value as if the volatility was the *higher* $\bar{\sigma}(z) \geq \sigma_\star(z)$ (where equality holds only if the fast volatility factor was actually constant).

6 Examples, Numerical Solutions and Accuracy of Approximation

We first present the common family of power utilities, for which there are explicit solutions in the constant Sharpe ratio case. Then, in Section 6.2, we introduce a family of mixture of power utility functions that allow for declining risk-aversion with increasing wealth. We demonstrate that the asymptotic approximation can be computed numerically in an efficient manner even when there is no explicit solution for the zeroth order problem, and show the effects of fast stochastic volatility at differing wealth and risk tolerance levels. In Section 6.3, we give a proof of accuracy in the case of power utility and one volatility factor (in Section 6.3.1 with a fast factor and in Section 6.3.2 with a slow factor), where the HJB equation can be reduced to a linear PDE problem by a distortion transformation, as recalled at the beginning of Section 6.3. Finally, in Section 6.4, we quantify the accuracy of our slow scale approximation in a one-factor stochastic volatility model for which there is an explicit solution under power utility, using parameter values obtained from data by Chacko and Viceira [2005].

6.1 Power Utility Case

The canonical example of a utility function on R^+ is the power (or CRRA) utility:

$$U(x) = c\frac{x^{1-\gamma}}{1 - \gamma}, \qquad c, \gamma > 0, \quad \gamma \neq 1 \tag{69}$$

where γ is the coefficient of risk-aversion, and c is a weight for use later. Then its Arrow-Pratt measure of relative risk aversion is:

$$\mathrm{AP}[U] := -x\frac{U''(x)}{U'(x)} = \gamma,$$

and the risk-tolerance function at the terminal time T is

$$R(T, x) = -\frac{U'}{U''} = \frac{1}{\gamma}x. \tag{70}$$

It is well-known that the constant Sharpe ratio value function is given by

$$M(t, x; \lambda) = c\frac{x^{1-\gamma}}{1 - \gamma}g(t; \lambda), \qquad g(t; \lambda) = \exp\left(\frac{1}{2}\lambda^2\left(\frac{1 - \gamma}{\gamma}\right)(T - t)\right), \tag{71}$$

and, the risk-tolerance function $-M_x/M_{xx} = x/\gamma$ is independent of λ.

From (61), $v^{(0)}$ in the multiscale volatility approximation is given by $v^{(0)}(t, x, z) = M(t, x; \overline{\lambda}(z))$. In the special power case, the solution of Black's equation (29) with terminal condition (70) is given by $R(t, x; \lambda) = x/\gamma$, which is independent of t and λ. Therefore, we have $D_k = (x/\gamma)^k \frac{\partial^k}{\partial x^k}$. Consequently, the order ($\sqrt{\varepsilon}$, $\sqrt{\delta}$) approximation $\widetilde{V^{\varepsilon,\delta}}$ in (65) is explicitly given by

$$\widetilde{V^{\varepsilon,\delta}}(t, x, z) = \left(1 - \frac{1}{2}\rho_1\sqrt{\varepsilon}(T - t)B(z) + \frac{1}{2}\sqrt{\delta}(T - t)^2\rho_2\widehat{\lambda}(z)\overline{\lambda}(z)\overline{\lambda}'(z)g(z)\right)\left(\frac{1 - \gamma}{\gamma}\right)^2 v^{(0)}(t, x, z).$$

Moreover, because for power utility we have $R_{xx} = 0$, we see from (32) that D_1 and D_2 commute, and therefore the terms in (66) with $(D_1 + D_2)$ in them are zero because $D_2 v^{(0)} = -D_1 v^{(0)}$. The portfolio policy approximation (66) is thus given in this case by

$$\widetilde{\pi^{\varepsilon,\delta}} = \left[\frac{\lambda(y, z)}{\sigma(y, z)} + \frac{1}{2}\sqrt{\varepsilon}\rho_1\left(\frac{1 - \gamma}{\gamma}\right)^2\frac{a(y)\theta_y(y, z)}{\sigma(y, z)} + \sqrt{\delta}\rho_2\left(\frac{1 - \gamma}{\gamma}\right)\frac{g(z)}{\sigma(y, z)}(T - t)\overline{\lambda}(z)\overline{\lambda}'(z)\right]\left(\frac{x}{\gamma}\right).$$

The first term is the moving Merton strategy following both the fast and slow factors. In the power utility case, the risk-tolerance function R does not depend on the Sharpe ratio, and so the average $\overline{\lambda}(z)$ does not show up in this term as it would in the case of more general utilities. The second and third correction terms come from the fast and slow factors respectively, they are proportional to ρ_1 and ρ_2 respectively, they highlight the component of "intertemporal hedging" from the correlated piece of those factors, and they affect the value function only at the next order in ε and δ as in the general case presented in Section 4.2.

6.2   Mixture of Power Utilities

We now introduce a family of utility functions that allows for nonlinear risk tolerance (or, equivalently, non-constant relative risk aversion). These are described by

$$U(x) = c_1\frac{x^{1-\gamma_1}}{1 - \gamma_1} + c_2\frac{x^{1-\gamma_2}}{1 - \gamma_2}, \qquad c_1, c_2 \geq 0, \qquad \gamma_1 \geq \gamma_2 > 0, \quad \gamma_{1,2} \neq 1.$$
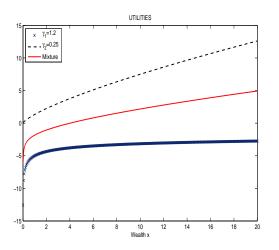
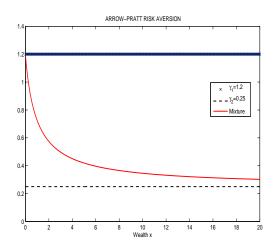Then we have the Arrow-Pratt measure of relative risk aversion:

$$\mathrm{AP}[U] = \frac{c_1\gamma_1 x^{-(\gamma_1-\gamma_2)} + c_2\gamma_2}{c_1 x^{-(\gamma_1-\gamma_2)} + c_2},$$

and its risk-tolerance function at time T is

$$R(T, x) = \left(\frac{c_1 x^{-(\gamma_1-\gamma_2)} + c_2}{c_1\gamma_1 x^{-(\gamma_1-\gamma_2)} + c_2\gamma_2}\right)x \sim \begin{cases} \frac{1}{\gamma_2}x & \text{as } x \to \infty \\ \frac{1}{\gamma_1}x & \text{as } x \to 0. \end{cases} \tag{72}$$

The mixture of two powers allows to mix unbounded above positive utilities (fractional powers with $\gamma < 1$) with unbounded below negative utilities ($\gamma > 1$). See Figures 1 and 2. How relative risk aversion varies

(a) Power utility functions $U_i = \frac{x^{1-\gamma_i}}{1-\gamma_i}$ and mixture $U = c_1 U_1 + c_2 U_2$ with $c_1 = c_2 = 1/2$.

(b) Arrow-Pratt risk-aversions $-xU''/U'$

Figure 1: Mixture of power utilities with $\gamma_1 = 1.2$ and $\gamma_2 = 0.25$.

with wealth is a subject of active empirical study, and we refer to Brunnermeier and Nagel [2008] and Liu et al. [2012] for some recent findings and debate. The mixture of two power utilities models declining risk aversion with increasing wealth, which one would naturally expect, and is supported by some of the empirical studies.

In the next subsection, we discuss how to solve the constant Sharpe ratio Merton problem numerically when there is no explicit solution.

6.2.1 Numerical Solution of Constant Sharpe Ratio Merton Problem

The first term in the approximations for the value function under either fast or slow or multifactor stochastic volatility is the Merton value function with a specific constant Sharpe ratio. Instead of solving the Merton PDE problem (10), we solve for the risk tolerance function $R(t, x; \lambda)$. In other words, we solve numerically the fast diffusion equation (29). Then we have

$$M_x(t, x; \lambda) = M_x(t, x_{\max}; \lambda) \exp\left(\int_x^{x_{\max}} \frac{1}{R(t, \xi; \lambda)} \, d\xi \right), \tag{73}$$

and

$$M(t, x; \lambda) = M(t, x_{\max}; \lambda) - \int_x^{x_{\max}} M_x(t, s; \lambda) \, ds, \tag{74}$$
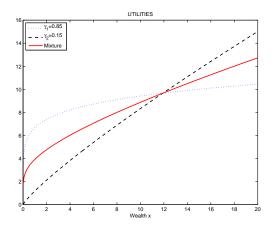
where $x_{\max}$ is large so that we can use large wealth asymptotics to insert $M_x(t, x_{\max}; \lambda)$ and $M(t, x_{\max}; \lambda)$.
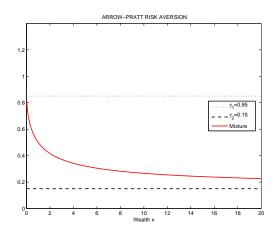
We solve the fast diffusion equation (29) with terminal condition (72) using implicit finite differences, viewing the PDE as linear with the diffusion coefficient frozen at the previous time step. On a $N \times J$ grid

$$\{(t_n, x_j) : t_n = T - n\Delta t, x_j = j\Delta x\} \quad \text{with} \quad \Delta t = \frac{T}{N}, \Delta x = \frac{x_{\max}}{J},$$

let $R_j^n \approx R(t_n, x_j; \lambda)$. The numerical scheme is

$$\frac{R_j^n - R_j^{n+1}}{\Delta t} + \frac{1}{2}\lambda^2 \left(R_j^n\right)^2 \frac{R_{j+1}^{n+1} - 2R_j^{n+1} + R_{j-1}^{n+1}}{(\Delta x)^2} = 0.$$

22

(a) Power utility functions $U_i = \frac{x^{1-\gamma_i}}{1-\gamma_i}$ and mixture $U = c_1 U_1 + c_2 U_2$ with $c_1 = c_2 = 1/2$.

(b) Arrow-Pratt risk-aversions $-xU''/U'$

Figure 2: Mixture of power utilities with $\gamma_1 = 0.85$ and $\gamma_2 = 0.15$.

Following (72) for the mixture of two power utilities, we use boundary conditions $R(t,0) = 0$ and $R_x(t, x_{max}) = \frac{1}{\gamma_2}$. Then we integrate according to (73) and (74) to find $v_x^{(0)}$ and $v^{(0)}$. For large wealth x, the investor behaves as if his risk-aversion is γ_2 and follows the corresponding Merton fixed-mix policy. As a consequence,

$$M \sim c_1 \frac{x^{1-\gamma_1}}{1-\gamma_1} g_{12}(t) + c_2 \frac{x^{1-\gamma_2}}{1-\gamma_2} g_2(t), \quad \text{as } x \to \infty,$$

where

$$g_2(t) = \exp\left(\frac{1}{2}\lambda^2 \left(\frac{1-\gamma_2}{\gamma_2}\right)(T-t)\right), \qquad g_{12}(t) = \exp\left(\frac{\lambda^2}{\gamma_2^2}(1-\gamma_1)\left(\gamma_2 - \frac{1}{2}\gamma_1\right)(T-t)\right).$$

Differentiating, gives $M_x \sim c_1 x^{-\gamma_1} g_{12}(t) + c_2 x^{-\gamma_2} g_2(t)$, as $x \to \infty$. These are used at large x_{max} in (73) and (74) to find M_x and M, which are plotted in Figures 3 and 4, and we observe that solving the fast diffusion equation numerically is tractable and efficient. These can then be used with the appropriate Sharpe ratio, $\overline{\lambda}$ for the fast, or $\lambda(z)$ for the slow or $\overline{\lambda}(z)$ for the multiscale case, to compute the zeroth order term $v^{(0)}$, as well as its partial derivatives which are needed in the higher order approximations.

6.2.2 Effect of Fast Stochastic Volatility Correction

We look at the effect of a fast volatility factor. Given the numerical solution of $v^{(0)}(t, x)$ for the mixture of power utilities with Sharpe ratio $\overline{\lambda}$, we numerically differentiate to find the fast scale correction (34) using the equivalent representation

$$v^{(1)}(t, x) = (T-t)\frac{1}{2}\rho_1 B(1 - R_x(t, x; \overline{\lambda}))R(t, x; \overline{\lambda})v_x^{(0)}(t, x).$$

The approximations to the value function at order zero and up to order $\sqrt{\varepsilon}$ are plotted in Figures 5(a) and 6(a). We can also represent the approximate indirect utilities $v^{(0)}(0, x)$ or $v^{(0)}(0, x) + \sqrt{\varepsilon} v^{(1)}(0, x)$ by their certainty equivalents (CEs) $U^{-1}(v^{(0)})$ and $U^{-1}(v^{(0)} + \sqrt{\varepsilon} v^{(1)})$, and these are shown in Figures 5(b) and 6(b).

Notice, as expected, accounting for the stochastic volatility correction lowers value functions and certainty equivalents. With the mixture of power utilities the impact of stochastic volatility is more at larger wealth levels where the investor's risk aversion is smaller.

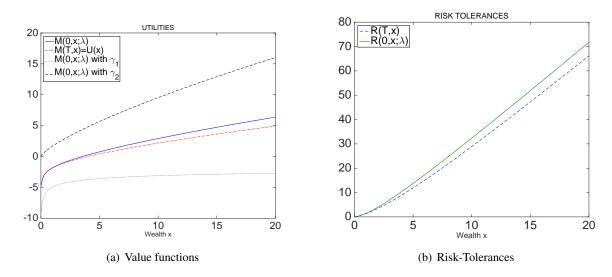(a) Value functions

(b) Risk-Tolerances

Figure 3: Constant parameter Merton value and risk tolerance functions for the mixture of power utilities with $\gamma_1 = 1.2$ and $\gamma_2 = 0.25$. Here $\lambda = 0.4$ and $T = 1$.

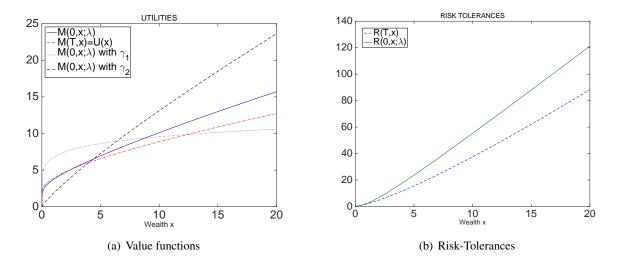(a) Value functions

(b) Risk-Tolerances

Figure 4: Constant parameter Merton value and risk tolerance functions for the mixture of power utilities with $\gamma_1 = 0.85$ and $\gamma_2 = 0.15$. Here $\lambda = 0.4$ and $T = 1$.
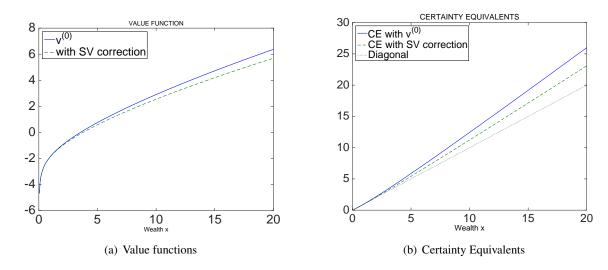
(a) Value functions

(b) Certainty Equivalents

Figure 5: Value function and certainty equivalents for the mixture of power utilities with $\gamma_1 = 1.2$ and $\gamma_2 = 0.25$, showing the zeroth order and the first two orders in the fast factor approximation. Here $\bar{\lambda} = 0.4$ and $T = 1$, and we plot the correction with $\sqrt{\varepsilon} \frac{1}{2} \rho_1 B = 0.01$.

(a) Value functions

(b) Certainty Equivalents
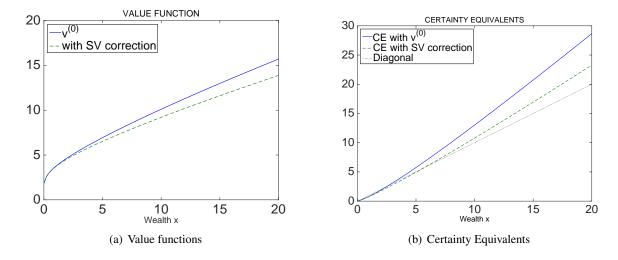
Figure 6: Value function and certainty equivalents for the mixture of power utilities with $\gamma_1 = 0.85$ and $\gamma_2 = 0.15$. Here $\bar{\lambda} = 0.4$ and $T = 1$, and we plot the correction with $\sqrt{\varepsilon} \frac{1}{2} \rho_1 B = 0.005$.

6.3 Accuracy of One-Factor Approximations under Power Utility

As is well-known, in models with only one stochastic volatility factor and when the utility function is of power type, the HJB equation can be reduced to a linear PDE by a distortion transformation. In this case, we give the proof of accuracy of our approximations to the value function in the fast and slow cases separately. However the linearization does not generalize to several volatility factors.

Specifically, we consider the one-factor model

$$dS_t = \mu(\xi_t)S_t \, dt + \sigma(\xi_t)S_t \, dW_t^{(0)} \tag{75}$$

$$d\xi_t = k(\xi_t) \, dt + h(\xi_t) \, dW_t^{(\xi)}, \tag{76}$$

where $W^{(0)}$ and $W^{(\xi)}$ are Brownian motions with instantaneous correlation coefficient between volatility and stock price shocks $\rho \in (-1, 1)$. The volatility factor ξ (driven by coefficients h and k) stands for either Y or Z with their corresponding coefficients. As derived in Zariphopoulou [2001], the value function at wealth level x under power utility (equation (69) with $c = 1$) is given by

$$V(t, x, \xi) = \frac{x^{1-\gamma}}{1 - \gamma} \Psi(t, \xi)^q, \tag{77}$$

where the distortion coefficient q is given by

$$q = \frac{\gamma}{\gamma + (1 - \gamma)\rho^2}, \tag{78}$$

and Ψ solves the linear PDE problem

$$\Psi_t + \left(\mathcal{L}_\xi + \frac{(1 - \gamma)}{\gamma} \lambda(\xi)\rho h(\xi)\frac{\partial}{\partial \xi} \right) \Psi + \frac{1}{2}\frac{(1 - \gamma)}{q\gamma} \lambda(\xi)^2 \Psi = 0, \qquad \Psi(T, \xi) = 1. \tag{79}$$

Here $\lambda(\xi) = \mu(\xi)/\sigma(\xi)$, and $\mathcal{L}_\xi$ is the generator of the process ξ:

$$\mathcal{L}_\xi = \frac{1}{2}h(\xi)^2 \frac{\partial^2}{\partial \xi^2} + k(\xi)\frac{\partial}{\partial \xi}.$$

We prove the accuracy of our approximation in this case of power utility, when the volatility factor is fast (Section 6.3.1) or slow (Section 6.3.2).

6.3.1 Fast Factor Accuracy

In the fast factor case where we replace ξ_t by Y_t in (3), we have $k(y) = \frac{1}{\varepsilon}b(y)$ and $h(y) = \frac{1}{\sqrt{\varepsilon}}a(y)$. From (19), (71), and Proposition 2.7, the fast volatility approximation to the value function for this case is given by:

$$V^\varepsilon(t, x, y) \approx v^{(0)}(t, x) + \sqrt{\varepsilon}v^{(1)}(t, x) = \frac{x^{1-\gamma}}{1 - \gamma}\left[1 - \sqrt{\varepsilon}(T - t)\frac{1}{2}\rho_1 B\left(\frac{1-\gamma}{\gamma}\right)^2 \right]\exp\left(\frac{1}{2}\bar{\lambda}^2\left(\frac{1-\gamma}{\gamma}\right)(T - t)\right), \tag{80}$$

with the effective Sharpe ratio $\bar{\lambda}$ defined in (17), and the constant B defined in (27).

In this section we provide a proof of accuracy of this approximation. The linear PDE (79) for $\Psi(t, y)$ becomes

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 \right)\Psi = 0, \qquad \Psi(T, y) = 1, \tag{81}$$

where $\mathcal{L}_0$ was defined in (6), and we define

$$\mathcal{L}_1 = -\Gamma\rho_1\lambda(y)a(y)\frac{\partial}{\partial y}, \qquad \mathcal{L}_2 = \frac{\partial}{\partial t} - \frac{\Gamma}{2q}\lambda(y)^2\cdot, \qquad \Gamma = \frac{\gamma - 1}{\gamma}, \tag{82}$$

and so from (78), we have $q = 1/(1 - \Gamma\rho^2)$ with $\Gamma < 1$.

This problem is now in the form of a singular perturbation problem of the type treated in Fouque et al. [2011], and the proof of accuracy follows the lines of the proof given there in Chapter 4, Section 5 in the case of a smooth terminal condition, which is what we have here with $\Psi(T, y) = 1$. We shall first show in Theorem 6.2 the accuracy of the approximation $\Psi(t, y) = \Psi_0(t) + \sqrt{\varepsilon}\Psi_1(t) + O(\varepsilon)$, where

$$\Psi_0(t) = \exp\left(-\frac{\Gamma\overline{\lambda}^2}{2q}(T - t)\right), \quad \Psi_1(t) = -(T - t)\left(\frac{\rho_1\Gamma^2 B}{2q}\right)\Psi_0(t). \tag{83}$$

Then, in Corollary 6.4, this is converted into a convergence result for the value function expansion.

The method consists of expanding $\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)\left(\Psi_0 + \sqrt{\varepsilon}\Psi_1 + \varepsilon\Psi_2 + \varepsilon^{3/2}\Psi_3 + \cdots\right)$ and choosing $\Psi_i (i = 0, 1, 2, 3)$ so that Equations (88)–(89) below are satisfied in order to cancel terms of order $\varepsilon^{-1}, \varepsilon^{-1/2}, \varepsilon^0$, and $\varepsilon^{1/2}$. This leads to the choices (83) above for Ψ_0 and Ψ_1, and (85) and (86) below for Ψ_2 and Ψ_3 respectively.

Assumption 6.1. We list here and comment on the assumptions we make on the class of models we are considering.

1. We assume $\gamma > 1$ corresponding to $\Gamma > 0$ in (82). We comment further on the case $\gamma < 1$ in Remark 6.3.

2. The second order linear differential operator $\mathcal{L}_0$ introduced in (6) is the infinitesimal generator of a one-dimensional diffusion process (in particular the coefficients $a(y)$ and $b(y)$ are at most linearly growing) which has a unique invariant distribution denoted by Φ in (4), is ergodic and has a spectral gap. This is as in Fouque et al. [2011], Chapter 3, Section 3, where it is also shown that this is the case for the commonly used OU and CIR processes.

3. The ergodic process with infinitesimal generator $\mathcal{L}_0$ admits moments of all order uniformly bounded in t (which is the case for OU and CIR processes).

4. Furthermore, we assume that $\lambda(y)$ is bounded, so that by Lemma 4.9 in Fouque et al. [2011], the diffusion process Y_t^ε with infinitesimal generator $\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1$, has moments of all order uniformly bounded in ε, that is, for any $k \in \mathbb{N}$, we have $E\{|Y_t^\varepsilon|^k\} \leq C_k(t, y)$ where $C_k(t, y)$ may depend on $(k, t < T, y)$ but not on $\varepsilon \leq 1$.

5. The source terms in the Poisson equations (24) and (87) being centered with respect to the invariant distribution Φ, by the Fredhlom alternative, solutions $\theta(y)$ for (24) and $\theta_1(y)$ for (87) exist and are characterized up to an additive constant (chosen freely). We further assume that $\theta(y)$ and $\theta_1(y)$ are at most polynomially growing. This is a weak assumption on the spectral properties of the operator $\mathcal{L}_0$ which is satisfied for OU and CIR processes as shown in Lemma 3.1 and Lemma 3.2 respectively in Fouque et al. [2011].

Theorem 6.2. *Under the assumptions listed above, for fixed $t < T$ and y, there is a constant C (which may depend on t and y) such that for any $\varepsilon \leq 1$:*

$$\left|\Psi(t, y) - \left(\Psi_0(t) + \sqrt{\varepsilon}\Psi_1(t)\right)\right| \leq C\varepsilon, \tag{84}$$

where the functions Ψ_0 and Ψ_1 are given in (83).

Proof. We define the functions $\Psi_i, i = 2, 3$ as follows:

$$\Psi_2(t, y) = \left(\frac{\Gamma}{2q}\right)\theta(y)\Psi_0(t), \tag{85}$$

where $\theta(y)$ is a solution to the Poisson equation (24) and the choice of $\theta(y)$ is up to a constant (in y) which does not play a role at this order of accuracy; and

$$\Psi_3(t, y) = \left(\frac{\rho_1\Gamma^2}{2q}\right)\theta_1(y)\Psi_0(t) - (T - t)\left(\frac{\rho_1\Gamma^3 B}{4q^2}\right)\theta(y)\Psi_0(t), \tag{86}$$

where $\theta_1(y)$ is a solution to the Poisson equation

$$\mathcal{L}_0\theta_1 = \lambda(y)a(y)\theta'(y) - \langle\lambda a\theta'\rangle, \tag{87}$$

defined up to a constant (in y) which does not play a role at this order of accuracy.

With this choice of functions $\Psi_i, i = 0, 1, 2, 3$, the following equations are satisfied:

$$\mathcal{L}_0\Psi_0 = 0, \tag{88}$$
$$\mathcal{L}_0\Psi_1 + \mathcal{L}_1\Psi_0 = 0,$$
$$\langle\mathcal{L}_1\Psi_1 + \mathcal{L}_2\Psi_0\rangle = \langle\mathcal{L}_2\rangle\Psi_0 = 0,$$
$$\mathcal{L}_0\Psi_2 + \mathcal{L}_1\Psi_1 + \mathcal{L}_2\Psi_0 = 0,$$
$$\langle\mathcal{L}_1\Psi_2 + \mathcal{L}_2\Psi_1\rangle = \langle\mathcal{L}_2\rangle\Psi_1 + \langle\mathcal{L}_1\Psi_2\rangle = 0,$$
$$\mathcal{L}_0\Psi_3 + \mathcal{L}_1\Psi_2 + \mathcal{L}_2\Psi_1 = 0. \tag{89}$$

Defining the residual

$$R^\varepsilon = \Psi - (\Psi_0 + \sqrt{\varepsilon}\Psi_1 + \varepsilon\Psi_2 + \varepsilon^{3/2}\Psi_3), \tag{90}$$

and using equation (81) satisfied by $\Psi(t, y)$ and equations (88)-(89), one obtains

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)R^\varepsilon = -\varepsilon(\mathcal{L}_1\Psi_3 + \mathcal{L}_2\Psi_2) - \varepsilon^{3/2}\mathcal{L}_2\Psi_3 \equiv \varepsilon S_\varepsilon(t, y), \tag{91}$$

where the source term $S_\varepsilon(t, y)$ can easily be computed using the expressions for $\mathcal{L}_1, \mathcal{L}_2, \Psi_2$, and Ψ_3. Using the terminal condition for Ψ in (81), and the terminal values for $\Psi_i, i = 0, 1, 2, 3$ found by setting $t = T$ in (83), (85), and (86), one obtains that the residual function R^ε satisfies the terminal condition

$$R^\varepsilon(T, y) = -\varepsilon\left(\frac{\Gamma}{2q}\right)\theta(y) + \varepsilon^{3/2}\left(\frac{\rho_1\Gamma^2}{2q}\right)\theta_1(y) \equiv \varepsilon T_\varepsilon(y). \tag{92}$$

Denoting by Y_t^ε the diffusion process with infinitesimal generator $\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1$, the residual R^ε, solution of the PDE problem (91)-(92), is given by the Feynman-Kac formula:

$$R^\varepsilon(t, y) = \varepsilon E\left\{e^{-\frac{\Gamma}{2q}\int_t^T \lambda^2(Y_s^\varepsilon)ds}T_\varepsilon(Y_T^\varepsilon) + \int_t^T e^{-\frac{\Gamma}{2q}\int_t^s \lambda^2(Y_u^\varepsilon)du}S_\varepsilon(s, Y_s^\varepsilon)ds \mid Y_t^\varepsilon = y\right\}. \tag{93}$$

Under our assumptions, one sees by direct computation that T_ε and S_ε are at most polynomially growing in y, and with $\Gamma > 0$, one obtains $|R^\varepsilon(t, y)| \le \varepsilon C$. Combined with (90), one deduces the desired accuracy estimate (84). $\qquad\square$

Remark 6.3. Under the assumption $\lambda(y)$ bounded, the condition $\Gamma > 0$ (or equivalently $\gamma > 1$) is not really needed as the exponential "discount" factors in (93) are uniformly bounded in ε. The assumption on $\lambda(y)$ can easily be relaxed for instance by assuming polynomial growth. In that case, either one assumes $\Gamma > 0$ and the exponential factors are still bounded (by 1), or else, if $\Gamma < 0$, a uniform bound for exponential moments of additive functionals of $\lambda^2(Y_s^\varepsilon)$ is needed. They are satisfied for instance in the case of an OU process and $\lambda(y) = y$ or in the case of a CIR process with $\lambda(y) = \sqrt{y}$.

Now we can undo the distortion transformation.

Corollary 6.4. *Under the assumptions of Theorem 6.2, for fixed (t, x, y), one has*

$$V^\varepsilon(t, x, y) = v^{(0)}(t, x) + \sqrt{\varepsilon} v^{(1)}(t, x) + O(\varepsilon),$$

where $v^{(0)}(t, x)$ and $v^{(1)}(t, x)$ are given explicitly in (80).

Proof. Using (77), (84), and the explicit formulas for Ψ_0 and Ψ_1 in (83), one obtains

$$V^\varepsilon(t, x, y) = \frac{x^{1-\gamma}}{1-\gamma}\left(\Psi_0(t) + \sqrt{\varepsilon}\Psi_1(t) + O(\varepsilon)\right)^q$$

$$= \frac{x^{1-\gamma}}{1-\gamma}\left(\Psi_0(t)^q + q\sqrt{\varepsilon}\Psi_1(t)\Psi_0(t)^{q-1} + O(\varepsilon)\right)$$

$$= \frac{x^{1-\gamma}}{1-\gamma}e^{-\frac{\Gamma\bar\lambda^2}{2}(T-t)}\left(1 - \frac{1}{2}\sqrt{\varepsilon}(T-t)\rho_1\Gamma^2 B\right) + O(\varepsilon),$$

which is (80) when taking into account that $\Gamma = \frac{\gamma-1}{\gamma}$ by (82). $\qquad\square$

6.3.2 Slow Factor Accuracy

In the slow factor case where we replace ξ_t by Z_t in (38), we have $k(z) = \delta c(z)$ and $h(z) = \sqrt{\delta}g(z)$. In this case, from (43) and Proposition 3.3, we have $V^\delta \approx v^{(0)} + \sqrt{\delta}v^{(1)}$, where

$$v^{(0)}(t, x, z) = \frac{x^{1-\gamma}}{1-\gamma}e^{-\frac{\Gamma\lambda(z)^2}{2}(T-t)}, \tag{94}$$

$$v^{(1)}(t, x, z) = \frac{1}{2}(T-t)^2\rho_2\Gamma^2\lambda(z)^2\lambda'(z)g(z)v^{(0)}(t, x). \tag{95}$$

In this section we provide a proof of accuracy of this approximation. The linear PDE (79) for $\Psi(t, z)$ becomes

$$\left(\delta\mathcal{M}_2 + \sqrt{\delta}\mathcal{M}_1 + \mathcal{L}_2\right)\Psi = 0, \qquad \Psi(T, z) = 1, \tag{96}$$

where $\mathcal{M}_2$ was defined in (40), and we define

$$\mathcal{M}_1 = -\Gamma\rho_2\lambda(z)\rho_2 g(z)\frac{\partial}{\partial z}, \qquad \mathcal{L}_2 = \frac{\partial}{\partial t} - \frac{\Gamma}{2q}\lambda(z)^2\cdot,$$

with the notation Γ introduced in (82).

This problem is now in the form of a regular perturbation problem of the type treated in Fouque et al. [2011], and the proof of accuracy follows the lines of the proof given there in Chapter 4, Section 5. We shall first show in Theorem 6.6 the accuracy of the approximation $\Psi(t, z) = \Psi_0(t, z) + \sqrt{\delta}\Psi_1(t, z) + O(\delta)$, where

$$\Psi_0(t, z) = \exp\left(-\frac{\Gamma\lambda(z)^2}{2q}(T-t)\right), \quad \Psi_1(t, z) = \left(\frac{\rho_2\Gamma^2}{2q}\right)(T-t)^2\lambda(z)^2\lambda'(z)g(z)\Psi_0(t). \tag{97}$$

Then in Corollary 6.8, this is converted into a convergence result for the value function expansion.

Assumption 6.5. We list here and comment on the assumptions we make on the class of models we are considering.

1. We assume that $\Gamma > 0$ (or equivalently $\gamma > 1$), and comment further on this assumption in Remark 6.7.

2. The second order linear differential operator $\mathcal{M}_2$ introduced in (40) is the infinitesimal generator of a one-dimensional diffusion process (in particular the coefficients $c(z)$ and $g(z)$ are at most linearly growing) which admits moments of all order uniformly bounded in $t \leq T$. Note that ergodicity for the slow factor Z is not required even though OU and CIR processes are commonly used in that case too.

3. We assume that $\lambda(z)$ is bounded and differentiable. In particular, by Lemma 4.9 in Fouque et al. [2011], the diffusion process Z_t^δ with infinitesimal generator $\delta \mathcal{M}_2 + \sqrt{\delta} \mathcal{M}_1$ has moments of all order uniformly bounded in δ for $t \leq T$, that is, for any $k \in \mathbf{N}$, we have $\sup_{t \leq T} \mathbf{E}\{|Z_t^\delta|^k\} \leq C_k(T, z)$ where $C_k(T, z)$ may depend on (k, T, z) but not on $\delta \leq 1$.

Theorem 6.6. *Under the assumptions listed above, for fixed $t < T$ and z, there is a constant C (which may depend on t and z) such that for any $\delta \leq 1$:*

$$\left| \Psi(t, z) - \left(\Psi_0(t, z) + \sqrt{\delta} \Psi_1(t, z) \right) \right| \leq C\delta, \tag{98}$$

where the functions Ψ_0 and Ψ_1 are given in (97).

Proof. With this choice of functions $\Psi_i, i = 0, 1$, the following equations are satisfied:

$$\mathcal{L}_2 \Psi_0 = 0, \tag{99}$$

$$\mathcal{L}_2 \Psi_1 + \mathcal{M}_1 \Psi_0 = 0. \tag{100}$$

Defining the residual

$$R^\delta = \Psi - (\Psi_0 + \sqrt{\delta} \Psi_1), \tag{101}$$

and using equation (96) satisfied by $\Psi(t, z)$ and equations (99)-(100), one obtains

$$\left(\delta \mathcal{M}_2 + \sqrt{\delta} \mathcal{M}_1 + \mathcal{L}_2 \right) R^\delta = -\delta \mathcal{M}_2 \Psi_0 - \delta^{3/2} \mathcal{M}_1 \Psi_1 - \delta^2 \mathcal{M}_2 \Psi_1 \equiv \delta S_\delta(t, z), \tag{102}$$

where the source term $S_\delta(t, z)$ can easily be computed using the expressions for $\mathcal{M}_1, \mathcal{M}_2, \Psi_0$, and Ψ_1. Using the terminal condition for Ψ in (96), and the terminal values for $\Psi_i, i = 0, 1$ given in (97), one obtains that the residual function R^δ satisfies the terminal condition

$$R^\delta(T, z) = 0. \tag{103}$$

Denoting by Z_t^δ the diffusion process with infinitesimal generator $\delta \mathcal{M}_2 + \sqrt{\delta} \mathcal{M}_1$, the residual R^δ, solution of the PDE problem (102)-(103), is given by the Feynman-Kac formula:

$$R^\delta(t, z) = \delta \mathbf{E} \left\{ \int_t^T e^{-\frac{\Gamma}{2q} \int_t^s \lambda^2(Z_u^\delta) du} S_\delta(s, Z_s^\delta) ds \mid Z_t^\delta = z \right\}. \tag{104}$$

Under our assumptions, one sees by direct computation that S_δ is at most polynomially growing in z, and with $\Gamma > 0$, one obtains $|R^\delta(t, y)| \leq \delta C$, which, by (101), is the desired accuracy estimate (98). $\square$

Remark 6.7. As in the case of fast scale (Remark 6.3), under the assumption $\lambda(z)$ bounded, the condition $\Gamma > 0$ (or equivalently $\gamma > 1$) is not really needed as the exponential "discount" factor in (104) is uniformly bounded in δ. The assumption on $\lambda(z)$ can easily be relaxed for instance by assuming polynomial growth. In that case, either one assumes $\Gamma > 0$ and the exponential factor is still bounded (by 1), or else, if $\Gamma < 0$, a uniform bound for exponential moments of additive functionals of $\lambda^2(Z_s^\delta)$ is needed. It is satisfied for instance in the case of an OU process and $\lambda(z) = z$ or in the case of a CIR process with $\lambda(z) = \sqrt{z}$, which is precisely the model discussed in Section 6.4.

Undoing the distortion transformation, we obtain:

Corollary 6.8. *Under the assumptions of Theorem 6.6, for fixed* (t, x, z)*, one has*

$$V^\delta(t, x, z) = v^{(0)}(t, x, z) + \sqrt{\delta}v^{(1)}(t, x, z) + O(\delta), \tag{105}$$

where $v^{(0)}(t, x)$ *and* $v^{(1)}(t, x)$ *are given explicitly below in* (94) *and* (95) *respectively.*

Proof. Using (77), (98), and the explicit formulas in (97) for Ψ_0 and Ψ_1, one obtains

$$
\begin{aligned}
V^\delta(t, x, z) &= \frac{x^{1-\gamma}}{1 - \gamma} \left(\Psi_0(t, z) + \sqrt{\delta}\Psi_1(t, z) + O(\delta) \right)^q \\
&= \frac{x^{1-\gamma}}{1 - \gamma} \left(\Psi_0(t, z)^q + q\sqrt{\delta}\Psi_1(t, z)\Psi_0(t, z)^{q-1} + O(\delta) \right) \\
&= \frac{x^{1-\gamma}}{1 - \gamma} e^{-\frac{\Gamma\lambda(z)^2}{2}(T-t)} \left(1 - \frac{1}{2}\sqrt{\delta}(T - t)^2 \rho_2 \Gamma^2 \lambda(z)^2 \lambda'(z) g(z) \right) + O(\delta),
\end{aligned}
$$

which establishes (105) with $v^{(0)}$ and $v^{(1)}$ identified by the formulas (94) and (95). $\qquad\square$

6.4 Comparison with an Explicit Solution

Staying within the one-factor stochastic volatility models (75)-(76), and under power utility, the coefficients $\mu(\xi)$ and $\sigma(\xi)$ in (75), and $h(\xi)$ and $k(\xi)$ in (76) can be chosen so that the linear PDE problem (79) admits an explicit solution. This can be achieved for instance by making the coefficients of the PDE (79) affine in ξ, in which case the PDE reduces to ODEs of Riccati-type. Kraft [2005] takes ξ_t to be a CIR process and $\mu(\xi) \propto \xi$, $\sigma(\xi) \propto \sqrt{\xi}$, so that $\lambda(\xi) \propto \sqrt{\xi}$, that is the Heston stochastic volatility model.

Here, as another example, to illustrate the performance of our approximation, we work with a model considered in Chacko and Viceira [2005] where the volatility factor is slowly varying according to their fit to low frequency data, as described in the quote from their paper in our Section 1. Accordingly, we will now denote ξ by Z and use our notation for the slow factor in (38). The model studied in Chacko and Viceira [2005] has

$$\mu(z) = \mu, \quad \sigma(z) = z^{-1/2}, \quad c(z) = m - z, \quad g(z) = \beta\sqrt{z},$$

that is

$$
\begin{aligned}
\frac{dS_t}{S_t} &= \mu \, dt + \sqrt{\frac{1}{Z_t}} \, dW_t^{(0)} \\
dZ_t &= \delta(m - Z_t) \, dt + \sqrt{\delta}\beta\sqrt{Z_t} \, dW_t^{(2)}.
\end{aligned}
\tag{106}
$$

They assume the standard Feller condition $\beta^2 < 2m$, which does not involve the time scale parameter δ. The process Z is referred to as the "instantaneous precision", and the Sharpe ratio is $\lambda(Z_t) = \mu\sqrt{Z_t}$.

In the paper Chacko and Viceira [2005], the authors derive explicit solutions for infinite horizon consumption problems, rather than the expected utility of terminal wealth problem we analyze here. However,

we derive the explicit formula for their model for this problem as follows. The equation (79) for $\Psi(t, z)$ becomes

$$\Psi_t + \frac{1}{2}\beta^2 \delta z \Psi_{zz} + \left(\delta(m - z) + \sqrt{\delta}\frac{(1 - \gamma)}{\gamma}\beta\mu\rho_2 z\right)\Psi_z + \frac{1}{2}\frac{(1 - \gamma)}{q\gamma}\mu^2 z \Psi = 0, \qquad \Psi(T, z) = 1.$$

This admits a solution

$$\Psi(t, z) = e^{A(T-t)z + \tilde{B}(T-t)},$$

where the function A satisfies the Riccati ODE

$$A' = \frac{1}{2}\delta\beta^2 A^2 + \sqrt{\delta}\frac{(1 - \gamma)}{\gamma}\beta\mu\rho_2 A - \delta A + \frac{1}{2}\frac{(1 - \gamma)}{q\gamma}\mu^2, \qquad A(0) = 0, \tag{107}$$

and the function $\tilde{B}$ solves $\tilde{B}' = \delta m A$, with $\tilde{B}(0) = 0$. For $\gamma > 1$, which we shall assume here, the quadratic right side of the Riccati equation has two real roots, which we denote by $a_{\pm}$, and we have

$$A(\tau) = a_- \frac{1 - e^{-\alpha\tau}}{1 - \frac{a_-}{a_+}e^{-\alpha\tau}}, \qquad \tilde{B}(\tau) = \delta m \left[a_-\tau - \frac{2}{\delta\beta^2}\log\left(\frac{1 - \frac{a_-}{a_+}e^{-\alpha\tau}}{1 - \frac{a_-}{a_+}}\right)\right],$$

where α is the square root of the discriminant of the quadratic.

Therefore, we have the explicit solution

$$V^\delta(t, x, y, z) = \frac{x^{1-\gamma}}{1 - \gamma}e^{qA(T-t)z + q\tilde{B}(T-t)},$$

where, as in (78), $q = \gamma/\left(\gamma + (1 - \gamma)\rho_2^2\right)$. The principal asymptotic approximation term can be obtained by setting $\delta = 0$ in (107):

$$v^{(0)}(t, x, z) = \frac{x^{1-\gamma}}{1 - \gamma}e^{\frac{(1-\gamma)}{2\gamma}\mu^2(T-t)z}.$$
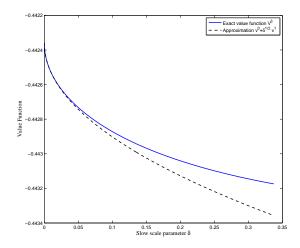
Then we have from (52),

$$v^{(1)}(t, x, z) = \frac{1}{2}(T - t)\rho_2\mu\beta z D_1 v_z^{(0)} = (T - t)^2\rho_2\beta\frac{(1 - \gamma)^2}{4\gamma^2}\mu^3 z v^{(0)}(t, x, z).$$

In Figure 7, we show the exact and approximate value function over a range of the time scale parameter δ up to the value estimated from monthly data in Chacko and Viceira [2005], $\delta = 0.3374$. At that value of δ, the relative absolute error in the approximation is 4.1×10^{-4}, which shows the approximation performs extremely well using real parameters from data. The figure also shows the portfolio weights (the fraction of wealth in the stock) using the exact formula and the order $\sqrt{\delta}$ approximation. At the largest value of δ, the relative absolute error in the approximation is 4.2%, which confirms the approximation error is small even at the level of the portfolio.

7 Conclusion

The impact of stochastic volatility on the problem of portfolio optimization can be studied and quantified through asymptotic approximations, which are tractable to compute. We have derived the first two terms of the approximations for the Merton value function, when volatility is driven by a single fast or slow factor, and Section 4 shows how these can be combined to incorporate both long and short time scales of volatility fluctuations. The methodology demonstrates progress that can be made in stochastic control problems in
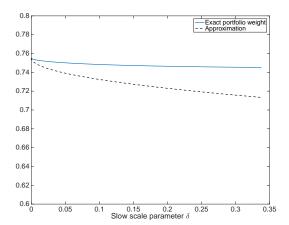
Figure 7: Exact and approximate value functions (left) and portfolio weights (right) in the slow scale volatility model (106) for a range of δ, and using the parameters estimated from data in Chacko and Viceira [2005]: $m = 27.9345, \rho_2 = 0.5241, \mu = 0.0811, \beta = 1.12$. We choose $\gamma = 3$, $z = Z_0 = m$ and $T = 2$. The last value of δ is the value estimated from data: $\delta = 0.3374$.

incomplete markets by viewing them as a perturbation around a complete markets problem which is well-understood.

There are a number of directions where similar techniques may play an effective role and we mention a few. First, the theory of forward utilities pioneered in Musiela and Zariphopoulou [2010] and related papers, where Black's fast-diffusion equation characterizes the evolution of utilities when volatility is constant. Second, problems where risk aversion is stochastic and correlated with market fluctuations is a natural issue when risk aversion and panic increases in market downturns.

The analysis here pertains to stochastic Sharpe ratio, which also includes stochastic predictability of asset returns, that is stochastic μ and constant volatility, as discussed for instance in Kim and Omberg [1996] and Wachter [2002]. Here, we have focused on the stochastic volatility interpretation because in the case μ constant, the asymptotic formulas can be related to quantities estimated from the implied volatility skew of option prices. A third direction would be to incorporate filtering of the stochastic predictability factor in a multiscale setting using asymptotic methods.

Finally, there is a long literature on the Merton problem under transaction costs, where asymptotic expansion in the cost parameter have been effective. We refer to the survey Guasoni and Muhle-Karbe [2013] for modern developments and background. The joint asymptotics to study the impact on portfolio choice of friction from both transaction costs and stochastic volatility is clearly of interest and a challenge.

A Using the Zeroth-Order Strategy in the Fast Factor Model

We have

$$\pi^{(0)}(t, x, y) = \frac{\lambda(y)}{\sigma(y)} R(t, x; \overline{\lambda}),$$

where R was defined in (11), and where now the wealth process X follows:

$$dX_t = \pi_t^{(0)} \mu(Y_t) \, dt + \pi_t^{(0)} \sigma(Y_t) \, dW_t^{(0)}.$$

33

The value of this strategy is $\tilde{V}^\varepsilon(t, x, y) = E\{U(X_T) \mid X_t = x, Y_t = y\}$, which solves the linear PDE

$$\tilde{V}_t^\varepsilon + \frac{1}{\varepsilon}\mathcal{L}_0\tilde{V}^\varepsilon + \frac{1}{2}\sigma(y)^2(\pi^{(0)})^2\tilde{V}_{xx}^\varepsilon + \pi^{(0)}\left(\mu(y)\tilde{V}_x^\varepsilon + \frac{\rho_1 a(y)\sigma(y)}{\sqrt{\varepsilon}}\tilde{V}_{xy}^\varepsilon\right) = 0,$$

where $\mathcal{L}_0$ was defined in (6), and with terminal condition $\tilde{V}^\varepsilon(T, x, y) = U(x)$.

The PDE can be re-written as:

$$\frac{1}{\varepsilon}\mathcal{L}_0\tilde{V}^\varepsilon + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1\tilde{V}^\varepsilon + \mathcal{L}_{t,x}(\lambda(y))\tilde{V}^\varepsilon = 0,$$

where $\mathcal{L}_{t,x}$ was defined in (13), and $\mathcal{L}_1 = \rho_1 a(y)\lambda(y)D_1\frac{\partial}{\partial y}$, with D_k as in (20).

Next we expand

$$\tilde{V}^\varepsilon(t, x, y) = \tilde{v}^{(0)}(t, x, y) + \sqrt{\varepsilon}\,\tilde{v}^{(1)}(t, x, y) + \varepsilon\tilde{v}^{(2)}(t, x, y) + \varepsilon^{3/2}\tilde{v}^{(3)}(t, x, y)\cdots,$$

and we will show that $\tilde{v}^{(0)} \equiv v^{(0)}$ and $\tilde{v}^{(1)} \equiv v^{(1)}$, and so $\tilde{V}^\varepsilon$ coincides with V^ε up to and including order $\sqrt{\varepsilon}$.

Inserting the expansion and comparing powers of ε, we find at order ε^{-1}: $\mathcal{L}_0\tilde{v}^{(0)} = 0$, and we satisfy this equation with $\tilde{v}^{(0)} = \tilde{v}^{(0)}(t, x)$, independent of y. At order $\varepsilon^{-1/2}$: $\mathcal{L}_0\tilde{v}^{(1)} + \mathcal{L}_1\tilde{v}^{(0)} = 0$, and as $\mathcal{L}_1\tilde{v}^{(0)} = 0$, we again choose $\tilde{v}^{(1)} = \tilde{v}^{(1)}(t, x)$, independent of y, to satisfy this equation. At the next order:

$$\mathcal{L}_0\tilde{v}^{(2)} + \mathcal{L}_1\tilde{v}^{(1)} + \mathcal{L}_{t,x}(\lambda(y))\tilde{v}^{(0)} = 0.$$

As $\mathcal{L}_1\tilde{v}^{(1)} = 0$, this is a Poisson equation for $\tilde{v}^{(2)}$ whose solvability condition is $\langle\mathcal{L}_{t,x}(\lambda(y))\tilde{v}^{(0)}\rangle = 0$, where the averaging $\langle\cdot\rangle$ was defined in (4). As $\tilde{v}^{(0)}$ doesn't depend on y, we have

$$\langle\mathcal{L}_{t,x}(\lambda(y))\tilde{v}^{(0)}\rangle = \langle\mathcal{L}_{t,x}(\lambda(y))\rangle\tilde{v}^{(0)} = \mathcal{L}_{t,x}(\overline{\lambda})\tilde{v}^{(0)}.$$

Expanding the terminal condition, we have $\tilde{v}^{(0)}(T, x) = U(x)$, which is the same as for $v^{(0)}$ in Section 2.2. From (21), $v^{(0)}$ satisfies the same PDE and terminal condition, and therefore $\tilde{v}^{(0)} \equiv v^{(0)}$.

Comparing terms of order $\sqrt{\varepsilon}$, we have

$$\mathcal{L}_0\tilde{v}^{(3)} + \mathcal{L}_1\tilde{v}^{(2)} + \mathcal{L}_{t,x}(\lambda(y))\tilde{v}^{(1)} = 0,$$

which is a Poisson equation for $\tilde{v}^{(3)}$ whose solvability condition is

$$\langle\mathcal{L}_1\tilde{v}^{(2)} + \mathcal{L}_{t,x}(\lambda(y))\tilde{v}^{(1)}\rangle = 0. \tag{108}$$

We know that

$$\mathcal{L}_0\tilde{v}^{(2)} = -\mathcal{L}_{t,x}(\lambda(y))\tilde{v}^{(0)} = -\mathcal{L}_{t,x}(\lambda(y))v^{(0)} = -\left(\mathcal{L}_{t,x}(\lambda(y)) - \mathcal{L}_{t,x}(\overline{\lambda})\right)v^{(0)} = -(\lambda(y)^2 - \overline{\lambda}^2)\left(\frac{1}{2}D_2 + D_1\right)v^{(0)},$$

and therefore

$$\tilde{v}^{(2)} = -\theta(y)\left(\frac{1}{2}D_2 + D_1\right)v^{(0)} + \tilde{C}(t, x),$$

where θ is a solution of the corrector equation (24), and $\tilde{C}$ is a constant (in y) of integration. Therefore (108) is $\mathcal{L}_{t,x}(\overline{\lambda})\tilde{v}^{(1)} = -\langle\mathcal{L}_1\tilde{v}^{(2)}\rangle = -BD_1D_2v^{(0)}$, where B was defined in (27), and we have used (28). The terminal condition is $\tilde{v}^{(1)}(T, x) = 0$, and so we have

$$\tilde{v}^{(1)}(t, x) = (T - t)BD_1D_2v^{(0)}(t, x) \equiv v^{(1)}(t, x),$$

from Proposition 2.7.

We conclude that using $\pi_t^{(0)}$ recovers the optimal value function up to order $\sqrt{\varepsilon}$.

B   Using the Zeroth-Order Strategy in the Slow Factor Model

We demonstrate that using the "moving Merton" zeroth order suboptimal strategy in the slow factor model $\pi^{(0)} = \frac{\lambda(z)}{\sigma(z)} R(t, x; \lambda(z))$ results in the optimal value up to first order $\sqrt{\delta}$, and so the corrections to the strategies impact the value function only at the $v^{(2)}$ term (order δ).

When volatility is slowly fluctuating as in model (38), we have the moving Merton policy

$$\pi^{(0)}(t, x, z) = \frac{\lambda(z)}{\sigma(z)} R(t, x; \lambda(z)),$$

where $R(t, x; \lambda(z))$ is as in (46) and the wealth process X now follows:

$$dX_t = \pi_t^{(0)} \mu(Z_t)\, dt + \pi_t^{(0)} \sigma(Z_t)\, dW_t^{(0)}.$$

The value of using this strategy is given by $\tilde{V}^\delta(t, x, z) = E\{U(X_T) \mid X_t = x, Z_t = z\}$. Then $\tilde{V}^\delta(t, x, z)$ solves the linear PDE

$$\tilde{V}_t^\delta + \delta \mathcal{M}_2 \tilde{V}^\delta + \frac{1}{2}\sigma(z)^2 (\pi^{(0)})^2 \tilde{V}_{xx}^\delta + \pi^{(0)}\left(\mu(z)\tilde{V}_x^\delta + \sqrt{\delta}\rho_2 g(z)\sigma(z)\tilde{V}_{xz}^\delta\right) = 0,$$

where $\mathcal{M}_2$ was defined in (40), and with terminal condition $\tilde{V}^\delta(T, x, z) = U(x)$.

The PDE can be re-written:

$$\mathcal{L}_{t,x}(\lambda(z))\tilde{V}^\delta + \sqrt{\delta}\mathcal{M}_1 \tilde{V}^\delta + \delta \mathcal{M}_2 \tilde{V}^\delta = 0,$$

with $\mathcal{L}_{t,x}(\lambda(z))$ given in (44), and $\mathcal{M}_1 = \rho_2 g(z)\lambda(z)D_1 \frac{\partial}{\partial z}$, with D_k is as in (45).

Next we expand

$$\tilde{V}^\delta(t, x, y) = \tilde{v}^{(0)}(t, x, z) + \sqrt{\delta}\,\tilde{v}^{(1)}(t, x, z) + \delta \tilde{v}^{(2)}(t, x, z) + \delta^{3/2}\tilde{v}^{(3)}(t, x, z)\cdots.$$

Inserting the expansion and comparing powers of δ gives

$$\mathcal{L}_{t,x}(\lambda(z))\tilde{v}^{(0)} = 0, \qquad \tilde{v}^{(0)}(T, x) = U(x),$$

and so $\tilde{v}^{(0)} \equiv v^{(0)}$ by uniqueness. At the next order,

$$\mathcal{L}_{t,x}(\lambda(z))\tilde{v}^{(1)} = -\mathcal{M}_1 \tilde{v}^{(0)}, \qquad \tilde{v}^{(1)}(T, x) = 0.$$

This is the same PDE as (51) for $v^{(1)}$ with the same terminal condition, and so again $\tilde{v}^{(1)} \equiv v^{(1)}$. We conclude that using $\pi_t^{(0)}$ recovers the optimal value function up to order $\sqrt{\delta}$.

C   Derivation of the Practical Strategy

We start with the HJB equation (56), but label the value function for the constrained optimization $\bar{V}^{\varepsilon,\delta}(t, x, y, z)$:

$$\bar{V}_t^{\varepsilon,\delta} + \left(\frac{1}{\varepsilon}\mathcal{L}_0 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3 + \delta \mathcal{M}_2\right)\bar{V}^{\varepsilon,\delta} + \text{NL}^{\varepsilon,\delta} = 0, \tag{109}$$

with $\bar{V}^{\varepsilon,\delta}(T, x, y, z) = U(x)$. Here, we have

$$\text{NL}^{\varepsilon,\delta} = \max_{\bar{\pi}}\left[\frac{1}{2}\sigma(y,z)^2\bar{\pi}^2\bar{V}_{xx}^{\varepsilon,\delta} + \bar{\pi}\left(\mu(y,z)\bar{V}_x^{\varepsilon,\delta} + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1\bar{V}^{\varepsilon,\delta} + \sqrt{\delta}\mathcal{M}_1\bar{V}^{\varepsilon,\delta}\right)\right]$$

where $\mathcal{L}_1$ and $\mathcal{M}_1$ are now defined as

$$\mathcal{L}_1 = \rho_1 a(y)\sigma(y,z)\frac{\partial^2}{\partial x \partial y}, \qquad \mathcal{M}_1 = \rho_2 \sigma(y,z)g(z)\frac{\partial^2}{\partial x \partial z}.$$

We look for an expansion of the function $\bar{V}^{\varepsilon,\delta}$, first in powers of $\sqrt{\delta}$:

$$\bar{V}^{\varepsilon,\delta} = \bar{V}^{\varepsilon,0} + \sqrt{\delta}\bar{V}^{\varepsilon,1} + \cdots,$$

and of the controls: $\bar{\pi} = \bar{\pi}^{(0,\varepsilon)} + \sqrt{\delta}\,\bar{\pi}^{(1,\varepsilon)} + \cdots$, where the principal terms $\bar{\pi}^{(0,\varepsilon)}$ and $\bar{\pi}^{(1,\varepsilon)}$ will be sought so as not to depend on y.

The equation for $\bar{V}^{\varepsilon,0}$ is obtained from (109) by setting $\delta = 0$:

$$\bar{V}_t^{\varepsilon,0} + \frac{1}{\varepsilon}\mathcal{L}_0\bar{V}^{\varepsilon,0} + \max_{\bar{\pi}^{(0,\varepsilon)}}\left(\frac{1}{2}\sigma(y,z)^2(\bar{\pi}^{(0,\varepsilon)})^2\bar{V}_{xx}^{\varepsilon,0} + \bar{\pi}^{(0,\varepsilon)}\mu(y,z)\bar{V}_x^{\varepsilon,0} + \frac{1}{\sqrt{\varepsilon}}\bar{\pi}^{(0,\varepsilon)}\mathcal{L}_1\bar{V}^{\varepsilon,0}\right) = 0.$$

Next, we expand $\bar{V}^{\varepsilon,0}$ as $\bar{V}^{\varepsilon,0} = \bar{v}^{(0)} + \sqrt{\varepsilon}\,\bar{v}^{(1,0)} + \varepsilon\bar{v}^{(2,0)} + \cdots$, and the control $\bar{\pi}^{(0,\varepsilon)} = \bar{\pi}^{(0)} + \sqrt{\varepsilon}\bar{\pi}^{(1,0)} + \cdots$.

Inserting the expansions and comparing powers of ε gives at order ε^{-1}: $\mathcal{L}_0\bar{v}^{(0)} = 0$, and we choose $\bar{v}^{(0)} = \bar{v}^{(0)}(t,x,z)$, independent of y, to satisfy this equation. At order $\varepsilon^{-1/2}$: $\mathcal{L}_0\bar{v}^{(1,0)} = 0$, and we again choose $\bar{v}^{(1,0)} = \bar{v}^{(1,0)}(t,x,z)$, independent of y, to satisfy this equation. At order one,

$$\bar{v}_t^{(0)} + \max_{\bar{\pi}^{(0)}}\left(\mathcal{L}_0\bar{v}^{(2,0)} + \frac{1}{2}\sigma(y,z)^2(\bar{\pi}^{(0)})^2\bar{v}_{xx}^{(0)} + \bar{\pi}^{(0)}\mu(y,z)\bar{v}_x^{(0)}\right) = 0. \tag{110}$$

For the maximizer $\bar{\pi}^{(0)}$ to not depend on y, the quantity being maximized must be y-independent, and so we choose $\bar{v}^{(2,0)}$ to be a solution of

$$\mathcal{L}_0\bar{v}^{(2,0)} + \left(\bar{v}_t^{(0)} + \frac{1}{2}\sigma(y,z)^2(\bar{\pi}^{(0)})^2\bar{v}_{xx}^{(0)} + \bar{\pi}^{(0)}\mu(y,z)\bar{v}_x^{(0)}\right) - \left(\bar{v}_t^{(0)} + \frac{1}{2}\bar{\sigma}(z)^2(\bar{\pi}^{(0)})^2\bar{v}_{xx}^{(0)} + \bar{\pi}^{(0)}\bar{\mu}(z)\bar{v}_x^{(0)}\right) = 0, \tag{111}$$

where $\bar{\sigma}(z)^2 = \langle\sigma(\cdot,z)^2\rangle$, and $\bar{\mu}(z) = \langle\mu(\cdot,z)\rangle$. so that the source of the Poisson equation (111) for $\bar{v}^{(2,0)}$ is centered and a solution exists.

With this choice of $\bar{v}^{(2,0)}$, (110) becomes

$$\max_{\bar{\pi}^{(0)}}\left(\bar{v}_t^{(0)} + \frac{1}{2}\bar{\sigma}(z)^2(\bar{\pi}^{(0)})^2\bar{v}_{xx}^{(0)} + \bar{\pi}^{(0)}\bar{\mu}(z)\bar{v}_x^{(0)}\right) = 0, \tag{112}$$

which is just the Merton PDE with separately fast-scale averaged σ^2 and μ:

$$\bar{v}_t^{(0)} - \frac{1}{2}\frac{\bar{\mu}(z)^2}{\bar{\sigma}(z)^2}\left(\frac{(\bar{v}_x^{(0)})^2}{\bar{v}_{xx}^{(0)}}\right) = 0, \qquad \bar{v}^{(0)}(T,x,z) = U(x).$$

Therefore, $\bar{v}^{(0)}(t,x,z) = M\left(t,x;\frac{\bar{\mu}(z)}{\bar{\sigma}(z)}\right)$.

From (112), we have:

$$\bar{\pi}^{(0)}(t,x,z) = -\frac{\bar{\mu}(z)}{\bar{\sigma}(z)^2}\frac{\bar{v}_x^{(0)}}{\bar{v}_{xx}^{(0)}} = \frac{\bar{\mu}(z)}{\bar{\sigma}(z)^2}R\left(t,x;\frac{\bar{\mu}(z)}{\bar{\sigma}(z)}\right).$$

References

Y. Aït-Sahalia and J. Jacod. *High-Frequency Financial Econometrics*. Princeton University Press, 2014.

F. Black. Investment and consumption through time. Reprinted as "Individual investment and consumption under uncertainty" in Luskin, D. L. (Ed.), Portfolio Insurance: A guide to dynamic hedging. Wiley, pp. 207-225., 1968.

M. Brunnermeier and S. Nagel. Do wealth fluctuations generate time-varying risk aversion? Micro-evidence on individuals' asset allocation. *American Economic Review*, 98(3):713–736, 2008.

G. Chacko and L. Viceira. Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets. *Review of Financial Studies*, 18:1369–1402, 2005.

J.-P. Fouque, G. Papanicolaou, and R. Sircar. *Derivatives in Financial Markets with Stochastic Volatility*. Cambridge University Press, 2000.

J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Sølna. *Multiscale Stochastic Volatility for Equity, Interest-Rate and Credit Derivatives*. Cambridge University Press, 2011.

P. Guasoni and J. Muhle-Karbe. Portfolio choice with transaction costs: a user's guide. In V. Henderson and R. Sircar, editors, *Paris-Princeton Lectures on Mathematical Finance 2013*. Springer, 2013.

M. Jonsson and R. Sircar. Partial hedging in a stochastic volatility environment. *Mathematical Finance*, 12(4):375–409, October 2002a.

M. Jonsson and R. Sircar. Optimal investment problems and volatility homogenization approximations. In A. Bourlioux, M. Gander, and G. Sabidussi, editors, *Modern Methods in Scientific Computing and Applications*, volume 75 of *NATO Science Series II*, pages 255–281. Kluwer, 2002b.

T. Kim and E. Omberg. Dynamic nonmyopic portfolio behavior. *Review of Financial Studies*, 9:141–161, 1996.

H. Kraft. Optimal portfolios and Heston's stochastic volatility model: an explicit solution for power utility. *Quantitative Finance*, 5:303–313, 2005.

X. Liu, F. Yang, and Z. Cai. Does relative risk aversion vary with wealth? Evidence from households' portfolio choice data. Technical report, University of North Carolina Charlotte, 2012.

R.C. Merton. *Continuous-Time Finance*. Blackwell, 1992.

M. Musiela and T. Zariphopoulou. Portfolio choice under space-time monotone performance criteria. *SIAM Journal on Financial Mathematics*, 1:326–365, 2010.

H. Pham. *Continuous-time Stochastic Control and Optimization with Financial Applications*. Springer, 2009.

R. Sircar and T. Zariphopoulou. Bounds & asymptotic approximations for utility prices when volatility is random. *SIAM J. Control & Optimization*, 43(4):1328–1353, 2005.

J. Wachter. Portfolio and consumption decisions under mean-reverting returns: An exact solution for complete markets. *Journal Financial and Quantitative Analysis*, 37(1):63–91, 2002.

T. Zariphopoulou. A solution approach to valuation with unhedgeable risks. *Finance and Stochastics*, 5:61–82, 2001.